TRANSFER OF ENERGY AND MATERIALS

IN FLUID SYSTEM

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TRANSFERS OF ENERGY AND MATERIAL IN FLUIDS

Foreword

In a large number of the situations with which an engineer is concerned in the process industries, transfers of energy, momentum and material in and between phases of flowing fluids are of importance. It behooves one interested in process development and design to become familiar with the fundamentals of such transfer processes in order that the ratesof transfer may be predicted with reasonable accuracy on the basis of available facts describing the situation.

The work of J. Boussinesq[1] and Osborne Reynolds[2] in the early part of the nineteenth century was one of the most outstanding early contributions to this subject. They laid the basis of most of the modern concepts of fluid mechanics and of the transfers of material and thermal energy in flowing fluids. The analogies between the transfer of momentum and the thermal transfer of energy were investigated and discussed by Reynolds[3], Prandtl[4], and Kármán[5], who made extensive contributions to the refinement of the analogy.

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[1] J. Boussinesq, Théorie de l' Ecoulement tourbillonant et tumulueux des liquides, Paris, (1897).

[2] Osborne Reynolds, "An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channels." Phil. Trans. Royal Soc., (1883). Also Collected Papers II.

[3] Osborne Reynolds, "On the Dynamical Theory of Incompressible Viscous Fluids and the determination of the Criterion." Phil. Trans. A, clxxxvi, 123 (1894).

[4] Prandtl, "Über Flüssigkeitsbewegung bei sehr kleiner Reibung," Verb. III Intern. Math. Congr., Heidelberg, (1904).

[5] von Kármán, "Mechanische Ähnlichkeit und Turbulenz," Nachr. Ges. Wiss., Göttingen, (1930).

In the first instance Reynolds considered that the turbulent core of a flowing fluid extended to the boundary of the conduit; whereas Prandtl postulated the existence of a laminar boundary layer wherein diffusional processes predominated. Von Kármán added, in a further modification, the concept of an intermediate transition layer where neither fully developed turbulent nor laminar flow obtained. It appears that in the course of time it will be possible to predict, with an accuracy comparable to that of most experimental work, the thermal transfers of energy to and through turbulently flowing streams for which the corresponding transfers of momentum are known.

Likewise the analogy between the transfer of material in and to turbulently flowing streams and the corresponding transfers of momentum has been pointed out by T. K. Sherwood[1]. However, the background of experimental information to test the veracity of such analogies is not nearly as extensive as that available for the study of the analogies between the thermal transfer of energy and the transfer of momentum discussed in the preceding paragraphs.

It is the purpose of this text to set forth some of the simpler concepts of fluid mechanics and their application to the description of the transfers of momentum associated with turbulently flowing fluid streams under conditions commonly encountered in industrial practice. Upon this background is then based a treatment of the thermal transfer of energy which is predicated largely upon the analogy to the transfer of momentum. Little attempt has been made to describe or even make reference to the large background of experimental facts concerning the thermal transfer of energy, since this subject has been covered in a satisfactory fashion by McAdams[2] in his text

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1 T. K. Sherwood, "Mass Transfer and Friction in Turbulent Flow". Trans. Am. Institute of Chem. Eng. 36, No. 6 817 (1940).

2 W. H. McAdams, "Heat Transmission", 2d Edition, McGraw-Hill Book Co., New York and London (1932).

entitled "Heat Transmission".

There is also included in this text a consideration of the transfer of materials associated with turbulently flowing fluid systems which treatment likewise is largely dependent upon the analogy with the transfer of momentum in similar systems.

This discussion has been prepared by the transfer group in the Chemical Engineering Laboratory. G. W. Billman, W. H. Corcoran, E. W. Hough and G. C. Standart were primarily responsible for the details of the material presented.

B. H. Sage

TABLE OF SYMBOLS

English Letters

A	Area of surface, ft.^{2}
	Gibbs' "psi" or Lewis' "maximum work" for a unit weight system, BTU/lb. $A = E - TS$
a	Acceleration, ft./sec.^{2}
b	Specific gas constant, Ft.^{3}/lb. · lb./in.^{2}/°R . $b = R/M$
C_g	Heat Capacity for a system of unit weight under specified conditions of restraint, BTU/lb.° F. $C_g = q_g/dT$
C_p	Isobaric heat capacity at constant composition for a system of unit weight, BTU/lb.° F. $C_p = \left(\frac{\partial H}{\partial T}\right)_{P,n} = q_{P,n}/dT$.
C_v	Isochoric heat capacity at constant composition for a system of unit weight, BTU/lb.° F. $C_v = \left(\frac{\partial E}{\partial T}\right)_{V,n} = q_{V,n}/dT$
D	Diameter of a circular section, ft.
D_k	Diffusivity of the kth component, ft.^{2}/sec.
D_o	Diameter of a circular conduit, ft.
E	Internal energy of a unit weight system, BTU/lb.
e	Height of a roughness element, ft.
F	Free energy of a unit weight system, (Gibbs' "zeta"), BTU/lb. $F = E + PV - TS$.
$\underline{F}$	Force, lb.
f	Fugacity, lb./in.^{2}.

$\underline{f}$	Fanning friction factor, dimensionless.
$\overset{+}{G}$	Weight velocity, lb./sec.ft.^{2}
g	Acceleration due to gravity, ft./sec.^{2}.
H	Enthalpy of a unit weight system, BTU/lb. $H = E + PV$
h	Heat transfer coefficient, BTU/ft.^{2}sec. $^\circ$F.
$\underline{h}$	Elevation above datum plane, ft.
J	Friction, BTU/lb.
j	An infinitesimal quantity of friction, BTU/lb.
k	Thermal conductivity, BTU/sec.ft.$^2 \cdot {}^\circ$F./ft.
	Kármán mixing length constant, dimensionless
L	Distance or length, ft.
l	Mixing length or characteristic length, ft.
	Latent heat of change of phase at constant temperature and composition for a system of unit weight, BTU/lb.
l_p	Latent heat of change of pressure at constant composition and temperature for a system of unit weight, BTU/lb./lb./in.^{2}
l_v	Latent heat of change of volume at constant composition and temperature for a system of unit weight, BTU/lb./ft.^{3}/lb.
lm_k	Latent heat of change in weight of component k at constant pressure, temperature, and weight of other components, BTU/lb.
$\underline{M}$	Molal weight or pound molecular weight, lb/lb. mole
m	Weight, lb.
$\underline{\underline{m}}$	Mass: lb.sec.^{2}/ft.

$\underline{m}$	Moles, lb. mole
n_g	Weight fraction of a system in the gas phase
n_k	Weight fraction of component k in the system as a whole
n_l	Weight fraction of a system in the liquid phase
$\underline{n}_k$	Mole fraction of component k in the system as a whole
P	Pressure, lb./in.^{2}
P'	Perimeter, ft.
Q	Heat associated with a change in state of a system of unit weight, BTU/lb.
q	Heat associated with an infinitesimal change in state of a system of unit weight, BTU/lb.
R	Molal gas constant, lb/in.^{2} - ft.^{3}/lb. mole/^{o}R
r	Polar radius in cylindrical and spherical coordinates, ft.
r_o	Radius of a circle or sphere, ft.
r_h	Mean hydraulic radius, ft.
r_d	Radius deficiency or distance from wall of circular conduit, ft.
r^+	Friction distance parameter for circular conduit, dimensionless.
S	Entropy of a system of unit weight, BTU/lb.^{o}R
T	Absolute temperature, ^{o}R
t	Temperature, ^{o}F.
U	Average velocity over a surface, ft./sec.
u	Velocity, ft./sec.
u_x, u_r, etc.	Velocity components in the X, R, etc. coordinate directions, ft./sec.

u_d	Velocity deficiency, ft./sec.
u_n	Velocity normal to an element of surface, ft./sec.
u_w	"Wall" velocity, ft./sec.
u_*	Friction velocity, ft./sec.
u^+	Velocity to friction velocity ratio, dimensionless
V	Specific volume, ft.^{3}/lb.
W	Work associated with a change of state of a system of unit weight, BTU/lb.
w	Work associated with an infinitesimal change in state of a system of unit weight, BTU/lb.
x	Coordinate distance in Cartesian system, axial distance in cylindrical coordinates, ft. (Usually called z in texts on mathematical physics)
y	Coordinate distance in Cartesian system, ft.
y_d	Deficiency distance or distance from wall of parallel plate conduit, ft.
y^+	Friction distance parameter, dimensionless.
Z	Compressibility factor, dimensionless $Z = \frac{PV}{bT}$
z	Coordinate distance in Cartesian system, ft.

Greek Letters

α	A constant, a parameter, an angle, etc.
β	Ditto
Γ	Weight rate of flow per unit perimeter, lb./sec.ft.

δ	Apparent laminar film thickness, ft.
ϵ	Turbulence factor or eddy viscosity, $lb./ft.^2$
	Infimitesimal velocity fluctuation, ft./sec.
ς	Rate of angular distortion, sec^{-1}
η	Absolute viscosity, $lb.sec./ft.^2$
θ	Time, sec.
κ	Thermometric conductivity or thermal diffusivity, $ft.^2/sec.$
λ	Mean free path, ft.
	Heat associated with non-equilibrium addition of material to a system of unit weight, BTU/lb.
	Resistance factor, dimensionless
λ_h	Resistance factor calculated by using mean hydraulic radius, dimensionless
μ	Chemical potential of a system of unit weight, BTU/lb.
ν	Kinematic viscosity, $ft.^2/sec.$
π	Similarily variable, dimensionless
ρ	Density or mass per unit of volume, $lb.sec.^2/ft.^4$
σ	Specific weight, $lb./ft.^3$
τ	Stress, $lb./in.^2$ (Shear stress when not otherwise specified).
τ_o	Shear stress at wall of conduit, $lb./in.^2$
τ_{xx}, etc.	Stress acting over element of surface perpendicular to X-axis, and acting in the X-direction, etc., $lb./in^2$.
τ_{xy}, etc.	Stress acting over element of surface perpendicular to X-axis and acting in Y-direction, etc., $lb./in.^2$
Φ	Acceleration caused by externalforces, $ft./sec.^2$

ϕ Azimuthal angle in spherical coordinates.

ψ Azimuthal angle in cylindrical coordinates, colatitude angle in spherical coordinates.

Joule-Thompson coefficient, °F./lb./in.^{2}

Ω External force potential, ft.^{2}/sec.^{2}.

ω Angular velocity, sec-^{1}.

ω' Vorticity, sec.-^{1}

General Modifications of Symbols for Properties

G The value of any property of a system.

G_A, G_B, etc. The value of the property of a system in state A, B, etc.

G_I, G_{II} The value of the property of system I, II, etc.

G_g The value of the property for material in the gas phase.

G_l The value of the property for material in the liquid phase.

G_k The value of the property for component k.

G' The value of the property when only one phase is present.

G'' The value of the property when two phases are present.

G_c The value of the property when the system is at the critical state.

G_R The reduced value of the property of a system or the value of the property divided by the value of the property when the system is at the critical state.

G_s The value of the property for the surroundings.

$\underset{\sim}{G}$ The residual value of the property.

$\bar{G}$ Time average of the value of the property.

$\underline{G}$	Total value of an extensive property.
G_i	Instantaneous value of the property of a system.
G_m	Maximum value of the property of a system.
$\dot{G}$	Time rate of change of the value of any property $G \equiv {}^{dG}/d\theta$
G_f	Fluctuation of the value of the property about the mean or time average value.
$\overset{+}{G}$	The rate of flow of an extensive property of a system through a surface, per unit area of surface.
$\overset{+}{\underline{G}}$	The rate of flow of an extensive property of a system through a surface.
$(G)_m$	The value of the property subject to the restriction that the weights of all components are constant.
$(G)_{m_i}$	The value of the property subject to the restriction that the weights of all components except component k are constant.
$(G)_{m_j}$	The value of the property subject to the restriction that the weights of all components except components k and n are constant.
$(G)_n$	The value of the property subject to the restriction that the weight fractions of all components are constant.
$(G)_{n_j}$	The value of the property subject to the restriction that the weight fractions of all components except components k and n are constant.

<u>Dimensionless Flow Variables</u>

Bi	Biot's Number
Ca	Cauchy's Number

Eu	Euler's Number
Fr	Froude's Number
Gr	Grashof's Number
Nu	Nusselt's Number
Pe	Peclet's Number
Re	Reynold's Number
St	Stanton's Number
We	Weber's Number

Mathematical Symbols

ln	Natural logarithm, to base e.
log	Common logarithm, to base 10.
ϕ or $\phi(x)$	Function or function of x.
Δ	Increment or difference operator.
Δ^2, etc.	Second difference operator, etc.
Δ_x, etc.	Partial forward difference operator with respect to x, etc,
$\Delta_{\bar{x}}$ etc.	Partial backward difference operator with respect to x, etc.
δ	Variation operator
d	Differential operator
∂	Partial differential operator
$\int dx$	Integral operator
$\int dL$	Line integral operator
$\oint dL$	Closed line integral operator ie line integral operator for closed path.
$\int_A dA$	Surface integral operator.

$\int_V dV$	Volume integral operator
$\sum$	Summation operator
∞	Symbol indicating that the variable exceeds any measurable bounds.
$\sim$	Same order of magnitude.
$\simeq$	Approximately equal.
$\cong$	Equal to a good approximation.

TABLE OF RESTRICTIONS ON APPLICATION OF EQUATIONS

In order to emphasize the region of application of each equation in the text as well as the limits to its validity, the proper restrictions are given in a square bracket [ ] just to the left of the equation number.

The meaning of the various symbols is as follows:

The symbol of a property (eg. P, V, S etc.) indicates constancy of that property.

q	heat is equal to zero
w	work is equal to zero
j	friction is equal to zero
r	reversible process
c	cyclic process
f	steady flow process
p	perfect gas
v	vander Waals gas
l	perfect liquid
i	idealized flow process
pp	flow between parallel plates
cc	flow in a cylindrical conduit
n	Newtonian fluid
h	homogeneous fluid
o	restrictions are involved but the reader is referred to the text for a statement

TABLE OF CONVERSION FACTORS

Since there are a few inconsistencies in the dimensions of the variables used in this discussion the following table is given to permit the conversion of all results to the pound, foot, second system of units or the BTU, pound, second system of units.

To convert from the units at the head of any row to the units at the head of any column multiply by the factor given in the table.

	lb./in.^{2}	lb./ft.^{2}	BTU/lb.	(lb./in.^{2})(ft.^{3}/lb.)
lb./in.^{2}	1	144	—	—
lb./ft.^{2}	0.006944	1	—	—
BTU/lb.	—	—	1	5.3993
(lb./in.^{2})(ft.^{3}/lb.)	—	—	0.18521	1

CHAPTER I

INTRODUCTION TO FLUID MECHANICS

1017-A-2

1-1 Fundamental Variables

In the discussion of flow problems in this course, the undefined quantities in terms of which all other physical variables will be expressed are force, length, time, and temperature. These quantities cannot themselves be expressed in terms of a smaller number of physical variables, although other independent quantities may be substituted for some or all of them. Thus in physics it is usual to take mass, length, time, and temperature as undefined. The reservation should be made that in statistical mechanics a variable can be constructed involving only mass, length, and time which can be successfully identified with the temperature of a system.

1-2 Frames of Reference

For various purposes, it may be desirable to view a flowing system from a position at rest with respect to part or all of the boundary surfaces, or at rest with respect to a part of the flow, or from a reference frame or coordinate system which moves in some other prescribed manner. Thus in a centrifugal pump, it may be desirable to view the flow from a position fixed in the casing so that both the rotor and the flow are in motion relative to the coordinate system, or it may be

CHAPTER I

INTRODUCTION TO FLUID MECHANICS

INTRODUCTION

Before considering the detailed mechanisms and the means of predicting the thermal transfers of energy to and through streams of flowing fluids, it is desirable to begin the analysis by discussing the simpler case of the isothermal flow of a fluid.

In this chapter a general background to fluid flow is given, and a number of relations for a particularily simple case of isothermal fluid flow are derived from the general principles and definitions set forth.

desirable to fix the frame of reference in the rotor or in a portion of the flow. Often such shifts of viewpoint introduce marked simplifications into the treatment of specific problems.

1-3 Definition of Element of Volume

Matter will be treated as continuous unless otherwise specified. That is, it will be assumed that the properties of the smallest portions into which we can conceive matter to be divided are the same as those of the substance in bulk, except for effects introduced by gradients.[1] Hence a small (strictly, an infinitesimal) portion of a flowing system can be bounded at a given instant by imaginary surfaces in such a manner that the values of all the physical properties, such as density, composition, etc. of the matter within the surfaces vary by only infinitesimal amounts from an average value. Such a small portion of the system will be called an element of volume. From this definition it follows that the dimensions of an element of volume may vary widely depending on the conditions of the problem and not all are necessarily infinitesimal.

1-4 Thermodynamics of Flowing Systems

It is assumed that the reader is familiar with Lacey and Sage Thermodynamics of One-Component Systems and Sage Thermodynamics of Multi-Component Systems; especially here, with chapters II and VII of the former book.

The systems usually considered by Gibbs[2] in thermodynamical treatments were those at equilibrium or those whose properties differed from the

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1 Lamb Hydrodynamics 6th Edition p-1, Dover (1945)

2 Collected Works, Volume I, 1931 Longmans, Green and Company.

equilibrium values only by infinitesimal amounts. Thus the only states of a system[1] which were considered were those in which conditions within the system were constant, both with respect to the further passage of time at a given point and at all points within the system at a given instant, due allowance being made for the modifying effect of gravitational, centrifugal, electrical, and magnetic fields on the properties of the system. Thus a thin flat cell, containing a gas, rotating at constant angular velocity about an axis parallel to one of the long dimensions would under isothermal conditions come to thermodynamic equilibrium if the effect of electric, magnetic, and gravitational fields could be neglected, even though to an observer stationary with respect to the cell, there would be measurable pressure changes in going from the axis toward the outer ends.

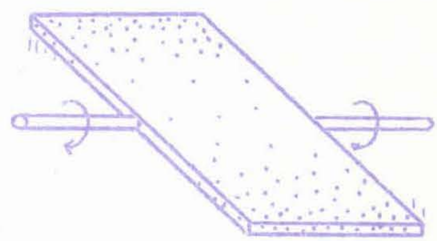

Fig. 1-1

The concepts of thermodynamics can often be extended, however, to cover the case of flowing systems by extending the concept of "state of a system" to that of the "state of a volume element"; that is, by describing the condition (ie giving the pressure, temperature, density, composition, etc.) of the matter in a volume element at a particular instant. By summing over all the volume elements in a given region, useful measures of extensive thermodynamic properties as well as of averages of intensive properties can be

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1 The state of a system is its condition at a particular instant. Lacey and Sage Thermodynamics of One Component Systems.

obtained. Some care is required to obtain <u>thermodynamic</u> properties in the volume element,however. In general such properties must be measured from a reference frame which moves with the volume element as the matter in it flows along, or other measured quantities must be corrected so as to give these properties. Thus the pressure measured when the instrument moves with the flow is often quite different from the apparent pressure measured by a stationary instrument. Likewise the apparent temperature will vary as the instrument moves with the flow, or remains fixed relative to the walls, in the latter case, the impact of the flow on the instrument will give a falsely high reading. <u>The terms pressure and temperature will only be used in the thermodynamic sense.</u>

Thermodynamics, alone, however, can give no information as to the rate at which processes occur in flowing systems,or any other kind of system for that matter.

1-5 <u>Stress and Deformation</u>

All materials are distorted or deformed to a greater or lesser extent by forces acting on them. In general the limiting value of the ratio of the force acting on a real or imaginary surface of the material to the area of the surface as the latter is diminished without limit is called the stress.

With some materials known as elastic solids the angular deformation is, at least approximately, proportional to the stress over a considerable range whether the forces are acting in compression or shear. Usually, also the proportionality constants are essentially the same whether the stress is being increased or decreased, except for inertia effects. Thus if τ

is the shearing stress or force per unit area being exerted tangentially on the illustrated (Fig 1-2) volume element by neighboring elements, $\frac{dx}{dy}$ is the angular deformation from the orginal position, then

$$\tau = k \frac{dx}{dy} \qquad [o] \qquad (1\text{-}1)$$

where k is the constant of proportionality known as the shear modulus.

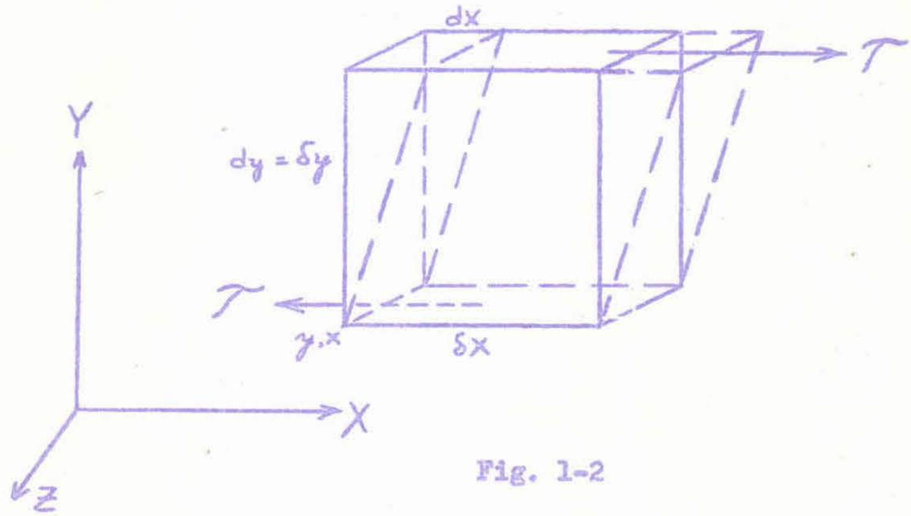

Fig. 1-2

The study of the deformation of elastic solids is called elasticity or strength of materials. Gibbs[1] and Goranson[2] discuss the thermodynamics of strained elastic solids. Crystals are the best examples of elastic solids.

With other materials, known as Newtonian fluids or simply fluids, the <u>rate</u> of angular deformation is strictly proportional to the shearing stress so that if the illustrated deformation is produced by the shear stress in the time interval $d\theta$,

$$\tau = \eta \frac{d}{dy} \frac{dx}{d\theta} = \eta \frac{dU}{dy} \qquad [n] \qquad (1\text{-}2)$$

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1 Collected Works, Longmans, Green and Company, 1931.
2 Thermodynamic Relations in Multicomponent Systems, Chapters V through IX, Carnegie Institute of Washington, 1930.

where η is the constant of proportionality known as the absolute viscosity; η is the same whether the stress is being increased or decreased, except for inertia effects. It can be seen that a fluid will not support a shearing stress at equilibrium. Gases and most liquids have been found experimentally to be Newtonian fluids.

The actual behavior of gases and liquids may often be satisfactorily approximated by that of perfect gases and incompressible liquids, respectively. Their relative deformations under pressure are:

for perfect gases

$$\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T = -\frac{1}{P} \qquad [p] \qquad (1\text{-}3)$$

while for incompressible liquids

$$\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T = 0 \qquad [1] \qquad (1\text{-}4)$$

Equation (1-4) for example may be considered to give the fractional change in volume for unit change in pressure.

There are also two other large groups of materials which do not come under either of the above classifications.

The non-Newtonian fluids, which are exemplified by colloidal solutions, have apparent viscosities which vary with the magnitude of shear stress and depend also on whether the stress is increasing or decreasing, even after discounting inertia effects, and plastic solids which deform like elastic solids under small stresses and flow like fluids under larger stresses which have exceeded a minimum value. The apparent elastic constants and apparent viscosity, however, depend on the magnitude of the stress and the direction of its change. Plastics, resins, pitches, clays, etc., are broadly classed as plastic solids.

There is no sharp demarcation between these various groups of materials which differ in degree and not in kind in their behavior, which is constrasted with the ideal behavior of Newtonian fluids and elastic solids in Fig. 1-3 and 1-4. (The curves may bend in the opposite sense in some cases).

A more complete classification and terminology is given in, " A Classification of Rheological Properties", Nature **149**, 702, (1942).

1-6 Types of Fluid Flow

There are two broad types of fluid flow-steady and unsteady. In the latter, the conditions at a given point in the flow change with time while in the former, the overall conditions at least, have achieved constant values with respect to time at each point throughout the flow.

Each of these types of fluid flow is further subdivided into laminar and turbulent flow. In the former, elements of volume of the fluid (generally in the shape of thin sheets or lamina) slip past one another without mixing so that the element, through deformed, never loses its identity. In the latter, however, in addition to the broad motion of the fluid, there are many secondary motions which can be considered more or less random in nature, by means of which all portions of the fluid are mixed and remixed.

In unsteady flow starting from rest, the motion is always laminar

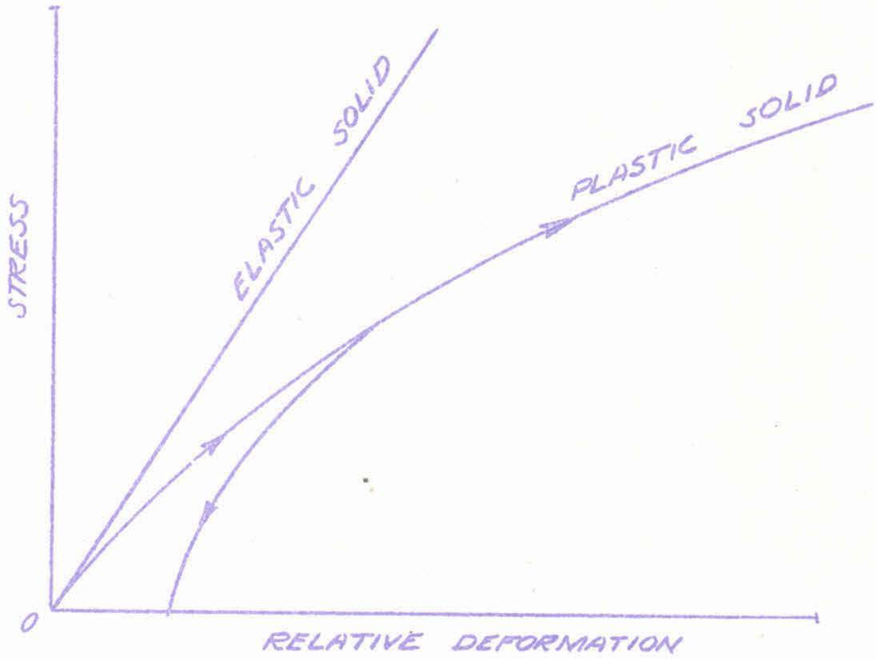
STRESS
ELASTIC SOLID
PLASTIC SOLID
0
RELATIVE DEFORMATION

Fig. 1-3

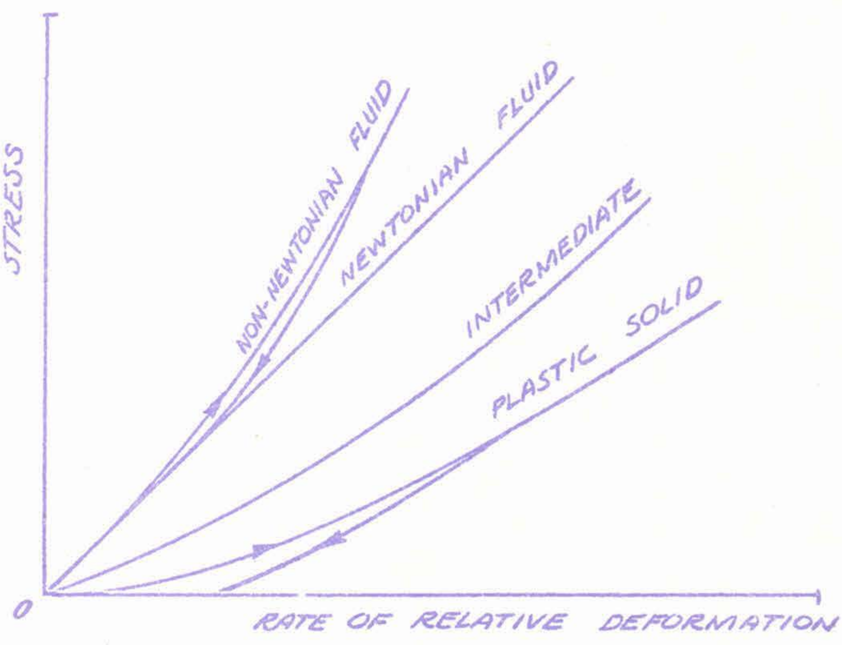
STRESS
NON-NEWTONIAN FLUID
NEWTONIAN FLUID
INTERMEDIATE
PLASTIC SOLID
0
RATE OF RELATIVE DEFORMATION

Fig. 1-4

initially, but as the velocity of flow increases a transition to turbulent flow will occur at sufficiently high velocities. This transition will also occur in many other ways, some of which involve steady flow systems.

1-7 Time Averages

In general the properties of a flowing system will vary from point to point even in steady laminar flow and with time as well at a given point in steady turbulent flow and in both types of unsteady motion.

Often the variations in the flow conditions resulting from turbulence are rather small compared with the overall motion, and it is useful to consider the instantaneous conditions as made up of an average component and a fluctuation. In unsteady turbulent motion, the general flow will itself be changing, but the concept of an average component will be useful if its time rate of change is small compared to the rate of change of the fluctuation so that a sensibly constant average condition exists, as far as the fluctuations are concerned, for a given short interval.[1] If these conditions are not met, the concept of turbulence will not be very useful e.g. in natural convection heat transfer.

Thus if G_i is the instantaneous value of an intensive property at a given point,

$$G_i = \bar{G} + G_f \tag{1-5}$$

where $\bar{G}$ is the time average of the property and G_f is the fluctuation value at time θ_o, and

$$\bar{G} = \frac{1}{\theta}\int_{\theta_o - \theta/2}^{\theta_o + \theta/2} G_i \, d\theta \tag{1-6}$$

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1 In general if no emphasis is intended, the word "rate" will henceforth be used instead of "time rate".

The time interval θ over which the averaging is done is long enough that $\bar{G}(\theta) \cong \bar{G}(\theta')$ for any $\theta' > \theta$ i.e. for a long enough time that the average value does not change significantly for longer times. It follows that

$$\bar{G}_f = 0 \tag{1-7}$$

For simplicity, when the averaging process is not being emphasized, the bar over the symbol for the value of the property will be omitted so that

$$G \equiv \bar{G} \tag{1-8}$$

and the word "average" will be used instead of "time average".

1-8 Flow Equation for an Important Special Case

Before developing the general material and energy flow equations, it is desirable to derive a number of special equations which will be of importance in future work.

Consider the case of the steady isothermal flow in a straight horizontal smooth conduit of constant cross section of an incompressible, homogeneous Newtonian fluid whose intensive properties are independent of pressure. If flow in such a conduit is considered at a point whose distance from the ends is sufficiently large compared to the hydraulic radius[1], the flow characteristics will depend only on conditions within the conduit. Such a flow process will be termed idealized flow. Under these circumstances, there can be no average flow perpendicular to the main stream and thus no average

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[1] The hydraulic radius is the ratio of the transverse cross-sectional area to the transverse wetted perimeter.

normal accelerations. Therefore, there can be no average pressure change along any path perpendicular to the flow, other than that due to external forces such as that of gravity, which will be neglected. These statements may be proved from the general equations of the flow to be developed later, and the magnitudes of the deviations from uniformity for more general cases can be calculated. Since the temperature is assumed constant, and the fluid homogeneous, all the other intensive properties of the fluid will be constant. Also since the flow is steady and the fluid incompressible there can be no acceleration in the direction of the flow.

Two of the most important straight conduits of constant cross section are the circular pipe and infinite parallel plates, and the following discussion will be restricted to them.

1-9 Flow in a Circular Pipe

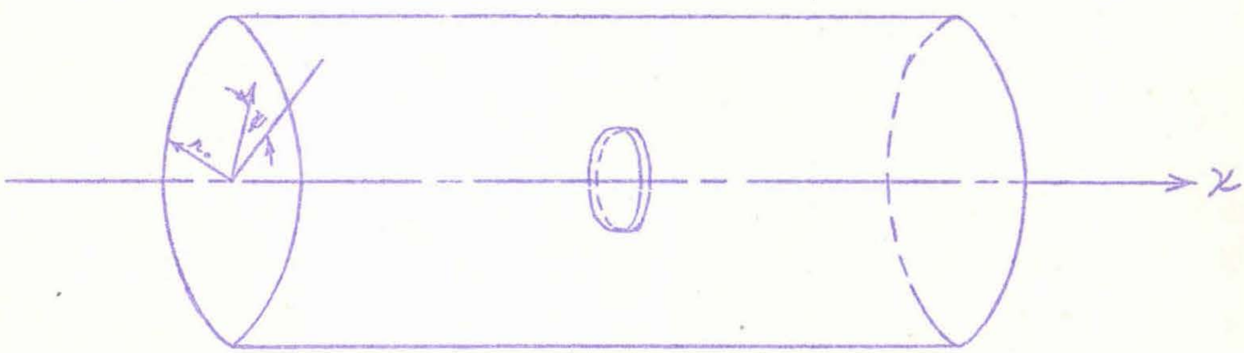

Fig. 1-5

Consider a straight circular pipe of radius r_o carrying fluid flowing under the above restrictions. The average rate of change of pressure along the pipe, $\frac{dP}{dx}$, also called the average pressure gradient, gives the difference in pressure between two cross sections at x and x + dx, per unit length

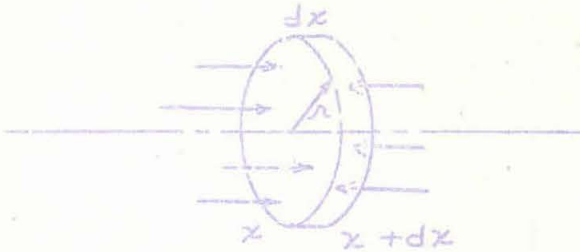

Fig. 1-6

Hence the average difference in force per unit length exerted over disks of radius r at x & x + dx is

$$-\pi r^2 \frac{dP}{dx} \qquad [1,cc]$$

This force difference per unit length produces no axial acceleration of the volume element of fluid shown and hence must be balanced by restraining shearing forces acting over the periphery of the element. Hence if τ is the average shearing stress (i.e. tangential force per unit area) over the periphery

$$2\pi r \tau dx = -\frac{dP}{dx}\pi r^2 dx \qquad [1,cc] \quad (1\text{-}9)$$

since from the symmetry of conditions τ must be constant for a given value of r and independent of the value of the polar angle ψ.

In follows that

$$\tau = -\frac{dP}{dx}\frac{r}{2} \qquad [1,cc] \quad (1\text{-}10)$$

and at the wall

$$\tau_o = -\frac{dP}{dx}\frac{r_o}{2} \qquad [i,cc] \qquad (1\text{-}11)$$

so

$$\frac{\tau}{\tau} = \frac{r}{r_o} \qquad [i,cc] \qquad (1\text{-}12)$$

Thus the shearing stresses are distributed linearly from the center of the conduit, independent of the type of flow.

The rate at which fluid is crossing the surface of the disk at x in Fig. (1-6) is, for any case in which the flow characteristics are independent of the polar angle ψ.

$$\dot{\underline{V}} = \int_o^r 2\pi r u_x dr = \pi r^2 U_x \qquad [s] \qquad (1\text{-}13)$$

where $\dot{\underline{V}}$ is the average volumetric time rate of flow, u_x the average flow in the x or axial direction at radius r, and U_x the average bulk velocity through the disk.

The total volumetric rate of flow is

$$\dot{\underline{V}}_o = \int_o^{r_o} 2\pi r u_x dr = \pi r_o^2 U_{x_o} = A U_{x_o} \qquad [s] \qquad (1\text{-}14)$$

where $\dot{\underline{V}}_o$ is the average total volumetric rate of flow, U_{x_o} the average total bulk velocity, and A the cross-sectional area of the conduit.

The average rate of dissipation of energy as friction[1] per unit length of flow is

$$\frac{\dot{\underline{j}}}{dx} = -\frac{dP}{dx}\dot{\underline{V}} \qquad [i,cc] \qquad (1\text{-}15)$$

in the element of radius r

and

$$\frac{\dot{\underline{j}}_o}{dx} = -\frac{dP}{dx}\dot{\underline{V}}_o \qquad [i,cc] \qquad (1\text{-}16)$$

for the flow as a whole.

Hence

$$\frac{\frac{\dot{\underline{j}}}{dx}}{\dot{\underline{V}}} = \frac{\frac{\dot{\underline{j}}_o}{dx}}{\dot{\underline{V}}_o} = -\frac{dP}{dx} \qquad [i,cc] \qquad (1\text{-}17)$$

It is assumed implicity that $\left(\frac{\partial E}{\partial P}\right)_T = 0$, which condition is very nearly satisfied for many liquids. Hence, the average energy dissipated as friction per unit volume, per unit length of flow is constant irrespective of the type of flow; or what is the same thing, $\frac{dP}{dx}$ is constant over a cross section.

This result can be expressed in another way. From Equations (1-10) and (1-13)

$$\frac{\dot{\underline{j}}}{dx} = -\frac{dP}{dx}\dot{\underline{V}} \quad \frac{2\tau}{r}\pi r^2 U_x = 2\pi r\tau U_x \qquad [i,cc] \qquad (1\text{-}18)$$

- -

1 For a discussion of the concept of friction see Lacey and Sage Thermodynamics of One Component Systems, Chapters II and VI.

and similarly

$$\frac{\dot{j}_o}{dx} = 2\pi r_o \tau_o U_{x_o} \qquad [i, cc] \qquad (1\text{-}19)$$

and τU_x is an average rate at which the shear stresses and the motion are dissipating energy as friction in the disk of radius r, etc.

1-10 <u>Flow between parallel flat plates</u>

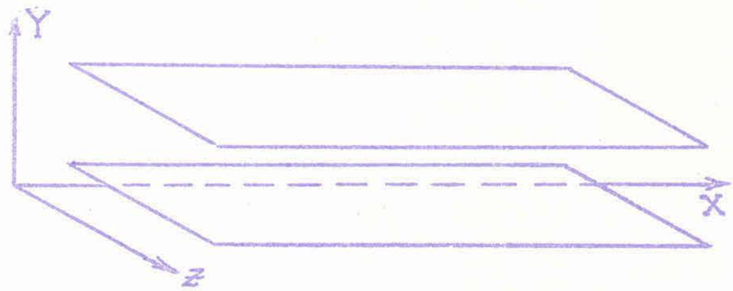

Fig. 1-7

In an exactly similar fashion, by considering the balance of the forces over a rectangular parallelepiped volume element of unit width in the z direction, thickness dx, depth dy,

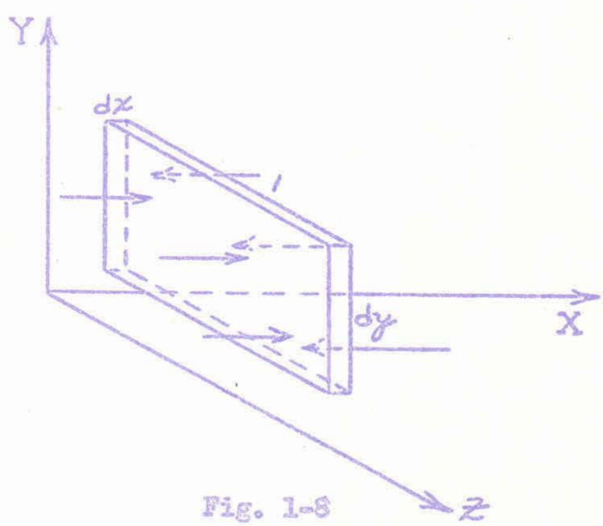

Fig. 1-8

the following equations are obtained

$$\tau = - y \frac{dP}{dx} \qquad [i,pp] \qquad (1\text{-}20)$$

$$\tau_o = - y_o \frac{dP}{dx} \qquad [i,pp] \qquad (1\text{-}21)$$

and

$$\frac{\tau}{\tau_o} = \frac{y}{y_o} \qquad [i,pp] \qquad (1\text{-}22)$$

Likewise

$$\underline{\mathring{V}} = 2\int_0^y u_x dy = 2yU_x \qquad [s] \qquad (1\text{-}23)$$

where $\underline{\mathring{V}}$ is the average volumetric rate of flow per unit width through the element of depth $2y$ and

$$\underline{\mathring{V}}_o = 2\int_0^{y_o} u_x dy = 2y_o U_{x_o} \qquad [s] \qquad (1\text{-}24)$$

where $\underline{\mathring{V}}_o$ is the average total volumetric flow rate per unit width.

Hence

$$\frac{\dot{j}}{dx} = -\frac{dP}{dx}\,\dot{V} \qquad [i,pp] \qquad (1\text{-}25)$$

$$\frac{\dot{j}_o}{dx} = -\frac{dP}{dx}\,\dot{V}_o \qquad [i,pp] \qquad (1\text{-}26)$$

as before

So

$$\frac{\frac{\dot{j}}{dx}}{\dot{V}} = \frac{\frac{\dot{j}_o}{dx}}{\dot{V}_o} = -\frac{dP}{dx} \qquad [i,pp] \qquad (1\text{-}27)$$

Also from Equation (1-20) and (1-23)

$$\frac{\dot{j}}{dx} = 2\tau U_x \qquad [i,pp] \qquad (1\text{-}28)$$

$$\frac{j_o}{dx} = 2\tau_o U_{x_o} \qquad [i,pp] \qquad (1\text{-}29)$$

give average rates at which the shear stresses and the motion are dissipating energy as friction in the volume elements.

1-11 <u>Laminar Flow in a Cylindrical Pipe</u>

From Equation (1-2) and (1-12) the shear stress on the periphery of an annular element of volume of radii r & $r + dr$ and length dx is

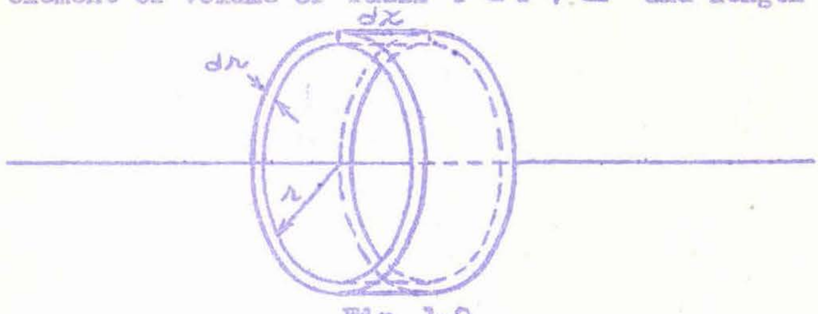

Fig. 1-9

$$\tau = \frac{\tau_o}{r_o} r = - \eta \frac{du_x}{dr} \qquad [1,lf,cc] \qquad (1\text{-}30)$$

since it is known from experiment that the flow is entirely in the axial direction, that is, that the fluid behaves as though the flow process consisted in the smooth slippage of cylindrical fluid layers past one another. Hence

$$u_x = C - \frac{\tau_o}{2\eta r_o} r^2 \qquad [1,lf,cc] \qquad (1\text{-}31)$$

where C is the constant of integration

Now when $r = 0$

$$u_x = u_{x_m} \qquad [1,cc] \qquad (1\text{-}32)$$

so

$$C = u_{x_m} \qquad [1,lf,cc] \qquad (1\text{-}33)$$

and

$$u_x = u_{x_m} - \frac{\tau_o}{2\eta r_o} r^2 \qquad [1,lf,cc] \qquad (1\text{-}34)$$

It is known, expecially because of the agreement of experiment with the present theory, that the velocity of a fluid in contact with a solid wall is zero. Hence when $r = r_o$,

$$u_x = 0 \tag{1-35}$$

and

$$u_{x_m} = \frac{\tau_o r_o}{2\eta} = -\frac{r_o^2}{4\eta}\frac{dP}{dx} \qquad [i,lf,cc] \tag{1-36}$$

from Equation (1-12) and (1-10). Hence

$$u_x = u_{x_m}\left(1 - \frac{r^2}{r_o^2}\right) \qquad [i,lf,cc] \tag{1-37}$$

and the velocity distribution is parabolic

$$\dot{V}_o = \int_0^{r_o} 2\pi r u_x dr = 2\pi u_{x_m}\int_0^{r_o} r\left(1 - \frac{r^2}{r_o^2}\right) dr = \frac{\pi r_o^2 u_{x_m}}{2} = \pi r_o^2 U_x \qquad [i,lf,cc] \tag{1-38}$$

Then

$$U_x = \frac{u_{x_m}}{2} \qquad [i,lf,cc] \tag{1-39}$$

Also from Equation (1-36) and (1-38)

$$-\frac{dP}{dx} = \frac{8\eta\dot{V}_o}{r_o^4} = \frac{8\eta U_x}{r_o^2} \qquad [i,lf,cc] \tag{1-40}$$

1-12 Laminar Flow Between Parallel Plates

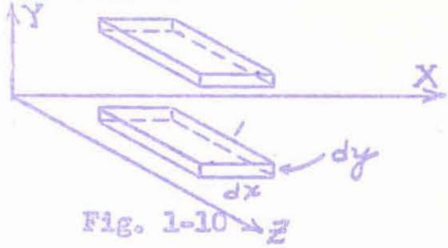

Fig. 1-10

As before from Equation (1-2) and (1-22)

$$\tau = \tau_o \frac{y}{y_o} = -\eta \frac{du_x}{dy} \qquad [i,lf,pp] \qquad (1\text{-}41)$$

by considering the two volume elements of unit width, length dx and depth dy at a distance y from the center plane.

Integration gives

$$u_x = C - \frac{\tau_o y^2}{2\eta y_o} \qquad [i,lf,pp] \qquad (1\text{-}42)$$

at $y = 0$

$$u_x = u_{x_m} \qquad [i,pp] \qquad (1\text{-}43)$$

so

$$u_x = u_{x_m} - \frac{\tau_o y^2}{2\eta y_o} \qquad [i,lf,pp] \qquad (1\text{-}44)$$

when $y = y_o$, $u_x = 0$ (1-45)

and from Equation (1-20)

$$u_{x_m} = \frac{\tau_o y_o}{2\eta} = -\frac{y_o^2}{2\eta}\frac{dP}{dx} \qquad [i,lf,pp] \qquad (1\text{-}46)$$

so

$$u_x = u_{x_m}\left(1 - \frac{y^2}{y_o^2}\right) \qquad [i,lf,pp] \qquad (1\text{-}47)$$

and the velocity distribution is again parabolic

The volumetric rate of flow per unit width is

$$\mathring{V}_o = 2\int_0^{y_o} u_x dy = 2u_{x_m}\int_0^{y_o}\left(1-\frac{y^2}{y_o^2}\right)dy = \frac{4u_{x_m}y_o}{3} = 2y_oU_x \qquad [i,lf,pp] \quad (1\text{-}48)$$

so

$$U_x = \frac{2}{3}u_{x_m} = -\frac{y_o^2}{3\eta}\frac{dP}{dx} \qquad [i,lf,pp] \quad (1\text{-}49)$$

or

$$-\frac{dP}{dx} = \frac{3\eta\mathring{V}_o}{2y_o^3} = \frac{3\eta U_x}{y_o^2} \qquad [i,lf,pp] \quad (1\text{-}50)$$

which is different from Equation (1-40).

1-13 Dimensionless Parameters

There is another simple approach to the problem of fluid motion which is especially useful for directing and correlating experimental work.

Although in the more complete treatment of this method of attack to be given later under Dimensional Analysis, a more fundamental approach is necessary, the correct relations can be obtained by simple if unrigorous reasoning.

Consider the same case as in Section 1-8 of a fluid flowing in a conduit with the same simplifying restrictions. The fluid motion can be, at least in part, described by the bulk velocity U, a length L perpendicular to the flow (the diameter of the pipe, for example), the viscosity η, density ρ, and pressure difference ΔP between two cross sections along the conduit. The inertia force of the flowing fluid or the product of the fluid mass and acceleration will be dimensionally equal to volume times density times velocity divided by time or $L^3\rho U/L/U = L^2\rho U^2$.

The viscous force is dimensionally equal to shearing stress times length squared or $\tau L^2 = \eta \frac{U}{L} L^2 = \eta UL$ (from Equation (1-2).

The forces causing dissipation of flow energy as friction are dimensionally equal to pressure difference times length squared or ΔPL^2

Presumably for the case under consideration, these quantities will determine the flow characteristics, and indeed any two ratios formed from them may describe the flow. It is customary to try the ratio of the inertia forces to the viscous forces, and the ratio of the dissipative forces to the inertia forces, the second to give one ratio not involving L and the first to avoid having ΔP appear in both ratios.

Hence if these ratios do describe the flow, there must be a relation connecting them and completely describing the fluid motion since the velocity, or pressure difference, or characteristic length is known experimentally to be determined if all the other variables are fixed.

Thus

$$\phi\left(\frac{\text{inertia forces}}{\text{viscous forces}}, \frac{\text{dissipative forces}}{\text{inertia forces}}\right) = 0 \qquad [1] \qquad (1\text{-}51)$$

or

$$\phi\left(\frac{LU\rho}{\eta}, \frac{\Delta P}{\rho U^2}\right) = 0 \qquad [1] \qquad (1\text{-}52)$$

$\frac{LU\rho}{\eta}$ will be recognized as Reynold's number, and $\frac{\Delta P}{\rho U^2}$ is known as Euler's number;

$$Eu = \frac{\Delta P}{\rho U^2} \qquad (1\text{-}53)$$

$$Re = \frac{LU\rho}{\eta} \qquad (1\text{-}54)$$

and

$$\phi\,(Re, Eu) = 0 \qquad [i] \qquad (1\text{-}55)$$

or

$$\frac{\Delta P}{\rho U^2} = \phi_1(R_e) \qquad [i] \qquad (1\text{-}56)$$

Euler's number is often modified slightly in case the pressure gradient, $\frac{dP}{dx}$ is known but not the pressure difference between two points. Then

$$Eu = -\frac{dP}{dx}\frac{L}{\rho U^2} \qquad (1\text{-}53a)$$

A large number of experiments have shown that there really is a function ϕ_1 which represents to a remarkably high degree of accuracy an enormous amount of data for widely varying conditions of fluid flow.(1)

1-14 Transition from Laminar to Turbulent Flow

It is found experimentally that the value of Reynold's number determines whether the flow is laminar or turbulent for this case; for circular pipes it is laminar for $Re < 2000$ and turbulent for $Re > 3000$

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(1) See Walker, Lewis, McAdams, and Gilliland, The Principles of Chemical Engineering, McGraw Hill Book Co. (1937), Fig. 27, p-78

approximately; for parallel plates the transition occurs at somewhat higher values[1]. Between the limits given, there is a transition region and the exact path followed depends on other factors than those considered above (e.g. on the shape of the entrance section to the conduit) so that there is no simple function ϕ_1 in this region. Further, by exercising special care, laminar flow may be extended beyond its normal limit for pipes to Reynold's numbers as high as 97,000[2]: any disturbance introduced into the flow, however, causes an immediate breakdown to turbulent motion throughout the channel.

The problem of the mechanism of the transition from laminar to turbulent flow is very difficult and has only very recently been solved to anything approaching a satisfactory degree of approximation by C. C. Lin[3]

1-15 Friction Factors

In engineering practice Euler's number (or ϕ_1(Re)) is identified with various friction factors. A great deal of confusion exists among the several factors, and care is necessary to ascertain just which one is being used by a particular author.

Fanning's friction factor f is defined as

$$f = \frac{1}{2}\,\phi_1(Re) \qquad [1] \qquad (1\text{-}57)$$

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1 (Goldstein Modern Development in Fluid Dynamics Vol. I p319). Oxford 1938.
2 Gibson Proc. Roy. Soc. Long. Vol. A83, p376, (1910).
3 Quarterly of Applied Mathematics Vol. III (1945) 117-142, 218-234; (1946) 277-301.

Another common friction factor which if often designated as λ is given by

$$\lambda = 2\phi_1(Re) \qquad [i] \qquad (1\text{-}58)$$

so

$$4f = \lambda \qquad [i] \qquad (1\text{-}59)$$

Another common approach utilizes the separate factors in the Euler's number ratio and defines the Fanning friction factor, for example, by (c.f. Equation 1-2)

$$\tau_o = f\frac{\rho U^2}{2} \qquad [i] \qquad (1\text{-}60)$$

From Equation (1-11)

$$-\frac{dP}{dx} = \frac{2\tau_o}{r_o} = \frac{4\tau_o}{D_o} = \frac{4f\rho U^2}{D_o \cdot 2} = \frac{\lambda\rho U^2}{D_o \cdot 2} \qquad [i, cc] \qquad (1\text{-}61)$$

and $Re = \dfrac{D_o U \rho}{\eta}$ $\qquad [i, cc] \qquad (1\text{-}62)$

for a circular pipe

For parallel plates, from Equation (1-21)

$$-\frac{dP}{dx} = \frac{\tau_o}{y_o} = \frac{2f\rho U^2}{2y_o \cdot 2} = \frac{1}{2}\frac{\lambda\rho U^2}{2y_o \cdot 2} \qquad [i, pp] \qquad (1\text{-}63)$$

and

$$Re = \frac{2y_o U\rho}{\eta} \qquad [i] \qquad (1\text{-}64)$$

since $L = 2y_o$. $\qquad (1\text{-}65)$

It can be seen that the formulas vary slightly with the shape of the cross section. In order to obtain uniform equations, and in the hope that a single variable could completely describe the effects of conduit shape, a mean hydraulic radius r_h was devised. This parameter is defined as the ratio of the transverse cross sectional area of flow to the transverse wetted perimeter.

Thus for a circular pipe

$$r_h = \frac{\frac{\pi}{4} D_o^2}{\pi D_o} = \frac{D_o}{4} \qquad (1\text{-}66)$$

Hence for all cases, Reynold's number is defined as

$$R_h = \frac{4 r_h U \rho}{\mu} \qquad [1] \qquad (1\text{-}67)$$

Equation (1-61) for pipes becomes,

$$-\frac{dP}{dx} = \frac{\tau_o}{r_h} = \frac{f_h \rho U^2}{r_h \cdot 2} = \frac{1}{4}\frac{\lambda_h \rho U^2}{r_h \cdot 2} \qquad [1] \qquad (1\text{-}68)$$

For parallel plates

$$r_h = \frac{1 \cdot 2y_o}{2 \cdot 1} = y_o \qquad [1, pp] \qquad (1\text{-}69)$$

so Equation (1-63) becomes

$$-\frac{dP}{dx} = \frac{\tau_o}{r_h} = \frac{f_h \rho U^2}{r_h \cdot 2} = \frac{1}{4}\frac{\lambda_h \rho U^2}{r_h \cdot 2} \qquad [1] \qquad (1\text{-}70)$$

identical with Equation (1-68)

$$Re_h = 2\,Re \qquad [i,pp] \qquad (1\text{-}71)$$

however.

Another useful relation is obtained from Eq. 1-68) or (1-70) by considering the second and fourth members

$$U = \sqrt{\frac{8}{\lambda_h}}\sqrt{\frac{\tau_o}{\rho}} \qquad [i] \qquad (1\text{-}72)$$

In the case of laminar flow these relations must agree with the analytic solutions already obtained.

For circular pipes, from Equation (1-40)

$$-\frac{dP}{dx} = \frac{32\eta U}{D_o^2} = \frac{32\eta \cdot U^2\rho}{D_oU\rho \cdot D_o} = \frac{32}{Re}\frac{U^2\rho}{D_o} \qquad [i,lf,cc] \qquad (1\text{-}73)$$

But from Eq. (1-61)

$$-\frac{dP}{dx} = 2f\cdot\frac{U^2\rho}{D_o} = \frac{\lambda}{2}\cdot\frac{U^2\rho}{D_o} \qquad [i,lf,cc] \qquad (1\text{-}61a)$$

So

$$\lambda = \frac{64}{Re}\ \&\ f = \frac{16}{Re} \qquad [i,lf,cc] \qquad (1\text{-}74)$$

For parallel plates, from Eq. (1-50)

$$-\frac{dP}{dx} = \frac{3\eta U}{y_o} = \frac{12\eta \cdot U^2\rho}{2y_oU\rho \cdot 2y_o} = \frac{12}{Re}\left(\frac{U^2\rho}{2y_o}\right) \qquad [i,lf,pp] \qquad (1\text{-}75)$$

and from Eq. (1-63)

$$f = \frac{12}{Re}\ \&\ \lambda = \frac{48}{Re} \qquad [i,lf,pp] \qquad (1\text{-}76)$$

Even if the hydraulic radius is employed

$$f_h = \frac{24}{Re_h} \quad \& \quad \lambda_h = \frac{96}{Re_h} \qquad [i, lf, pp] \qquad (1\text{-}77)$$

which equations are not identical with Equation (1-74).

In general exact analysis discloses that mean parameters designed to account for variation in shape do not completely succeed, although they may greatly improve the parallelism between similar flows.

A better approach[1], is to plot some exactly known flow parameter such as the numerical constants in Equations (1-74) and (1-77) against various shape parameters, such as the ratio of the shortest to the longest normal conduit dimension, for example, until a reasonably good correlation is obtained. New shapes, or irregular ones may be expected to follow the correlation.

Smooth and Rough Circular Pipes

The relationship expressed by Equation (1-56) for turbulent flow in smooth pipes was found by Blasius[2] from a compilation of existing data to be well represented by the empirical equation.

$$\lambda = 4f = 0.316/\sqrt[4]{Re} \qquad [i, tf, cc] \qquad (1\text{-}78)$$

An improved relation for the same conditions, advanced by Nikuradse[3] on the basis of more extensive data, is

$$\lambda = 0.0032 + \frac{0.221}{Re^{0.237}} \qquad [i, tf, cc] \qquad (1\text{-}79)$$

- -

1 Suggested by D. K. Barnes in a conversation with G. C. Standart.
2 ZdVDI 1912 also VDI Forsch. Arb. 131, 1913.
3 Nikuradse VDI Forsch. H. 356, 1932.

For turbulent flow in smooth pipes experimental velocity traverse of the conduit show that the velocity distribution is much flatter than the parabolic laminar distribution (Eq. (1-37) and (1-47)) in the center of the conduit, and much steeper near the walls. Thus (from Eq. (1-2)) there is little viscous action in the center of the conduit and almost all the viscous stresses are localized in the immediate vicinity of the walls where the flow is laminar.

Actual pipes are never perfectly smooth. Although the roughness is ordinarily negligible for drawn conduits, it is usually of considerable importance for those that are cast. As a beginning to a rational analysis, the irregular surfaces of real walls may be replaced by idealized rough surfaces consisting of a regular repetition of a unit protuberance. If a roughness element has a height e, for geometrically similar rough surfaces, an additional parameter not considered in the analysis beginning in Section 1-8 would be, for example e/r_o or r_o/e for pipes. Geometrically similar rough surfaces are defined to have a constant ratio of spacing of roughness element to height and geometrically similar shapes of protuberances.

Working with pipes coated with uniformly sized sand grains to secure constant surface conditions, Nikuradse[1] obtained data yielding plots of λ against Re for various relative smoothnesses, r_o/e (See Fig 1-10)

Such behavior can be explained on the basis of the flow characteristics. There is a definite relationship between λ and Re for smooth pipes which indicates that the turbulent motion does not depend on the nature of the walls of the pipe. As the relative smoothness decreases, the protuberances enter the laminar layer flowing next to the walls and at sufficiently high Reynold

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[1] VDI Forsch H 361, 1933.

numbers will project beyond this layer. In such circumstances, the flow is no longer influenced greatly by the laminar viscous shearing stresses since there is only a very thin laminar layer on the tops of the protuberances, which distort the flow pattern and introduce resistances overshading those resulting from the viscous resistances in the laminar layer. In such a case, the viscosity, and hence the Reynold's number as the flow parameter which contains viscosity, cannot influence the flow variables appreciably for a given roughness. Therefore, λ must be independent of Re and the λ curve parallel to the Re axis for a given roughness (or smoothness). For decreasing smoothness, the velocity at which the laminar layer will no longer be thick enough to cover the protuberances will decrease. Hence the transition from the "smooth" curve to the completely "rough" horizontal line will occur at decreasing Reynold's numbers.

L. F. Moody[1] has compiled the existing experimental data on friction factors for clean commercial pipes and plotted λ against Re for various values of e/D_o (relative roughness) as well as e/D_o against D_o for a wide range of commercial pipe types as is shown in Fig. 1-11 and 1-12. These two charts permit about as accurate a calculation of friction factors for clean commercial pipes as can be expected. It will be seen that the friction factor curves do not greatly resemble those for Nikuradse's data in their details. The most reasonable explanation for the failure of commercial pipe data to follow the smooth pipe curve, as the sand coated surfaces do, is to be found in the size distribution of protuberances in commercial pipe. Whereas the sanded pipe contains uniformly sized roughness elements which penetrate the laminar layer at about the same Reynold's number, the large

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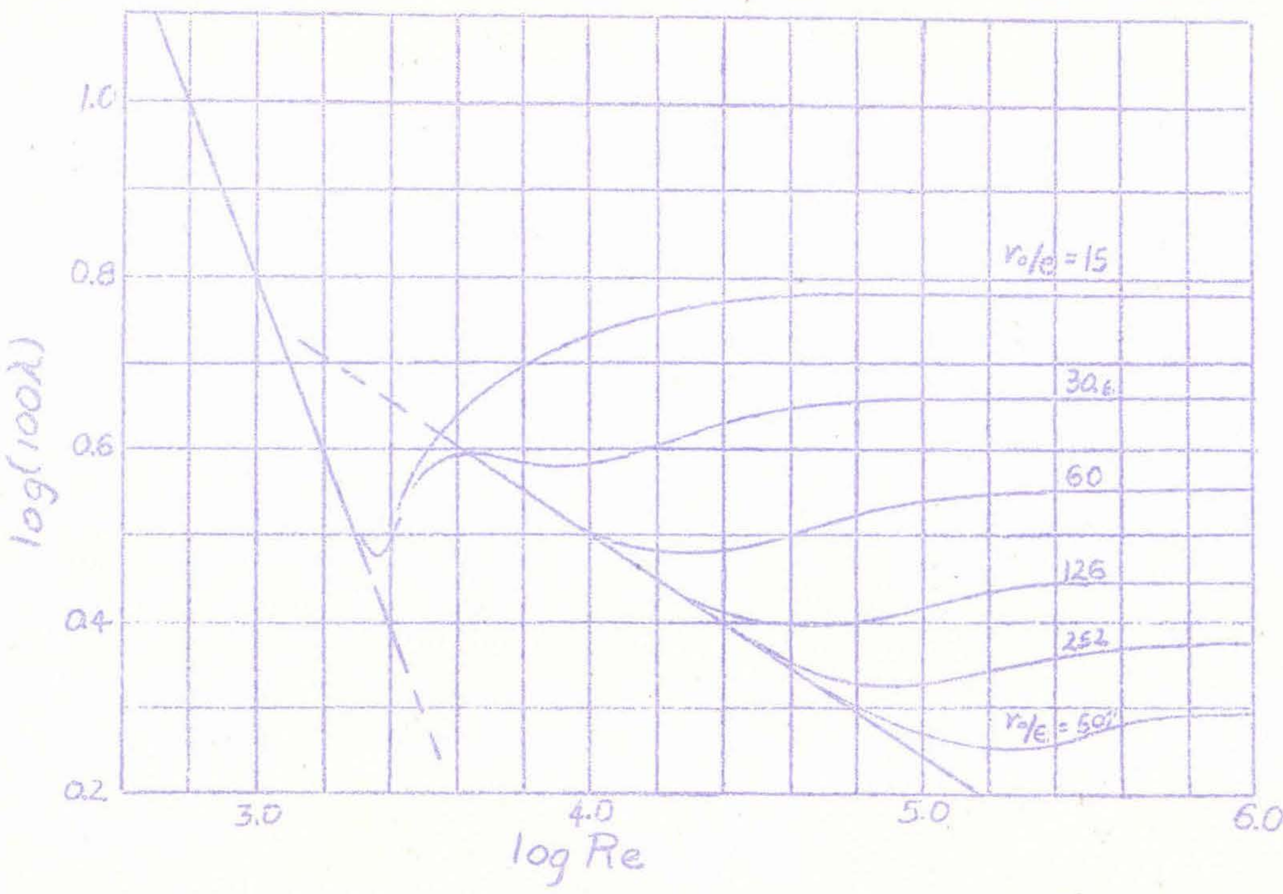
[1] Trans. ASME 66, (1944), 671-84.

size roughness elements of commercial pipe penetrate the film at any value above the critical value of 2,000. There is thus no case in which the laminar film masks all the protuberances and less tendency to follow the smooth pipe curve. Likewise, the small roughness elements are only gradually uncovered at increasing Re so that the transition to the rough pipe flow is more gradual also.

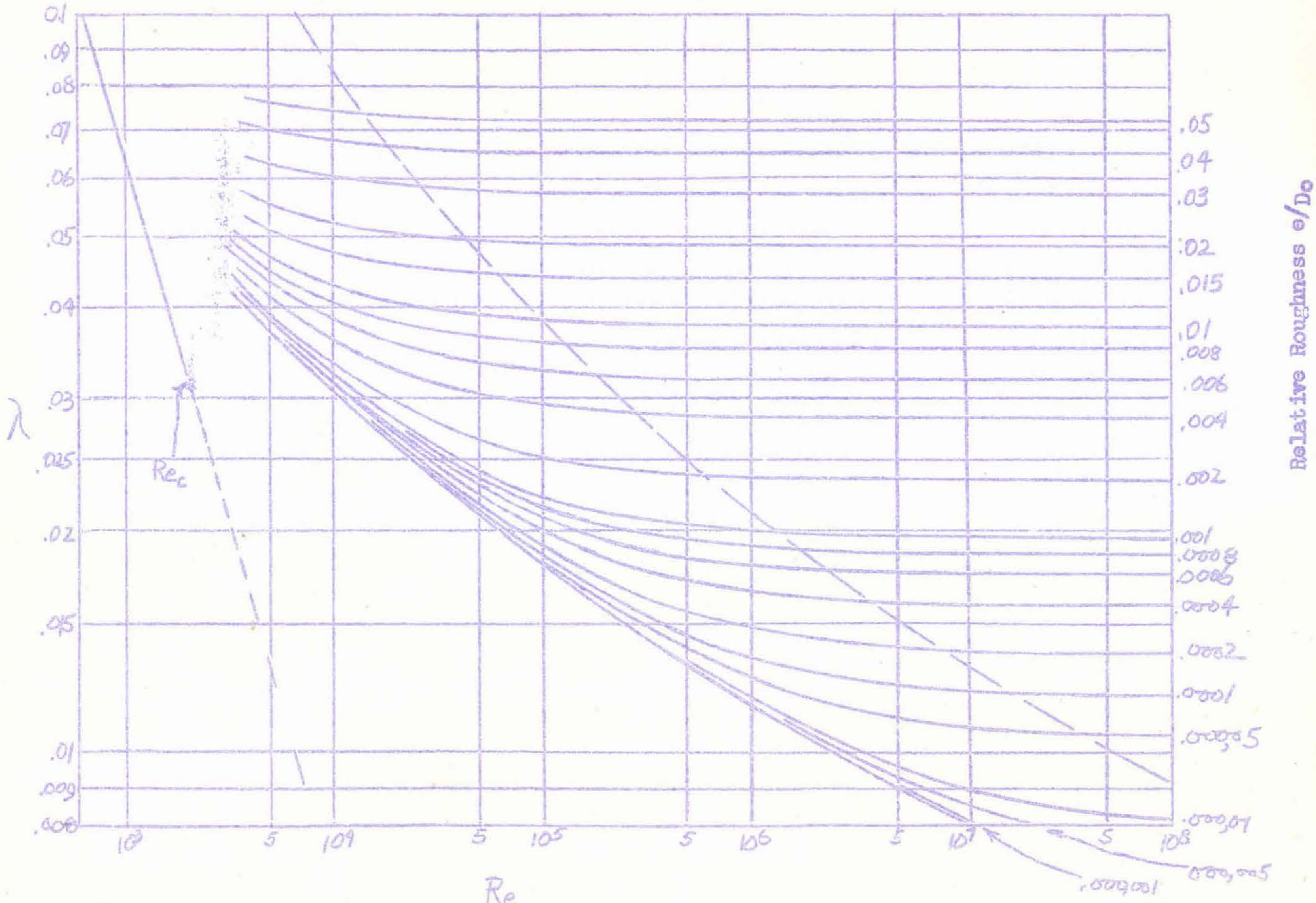


Friction Factor Chart for Smooth

and Rough Pipes

(after Nikuradse '33)

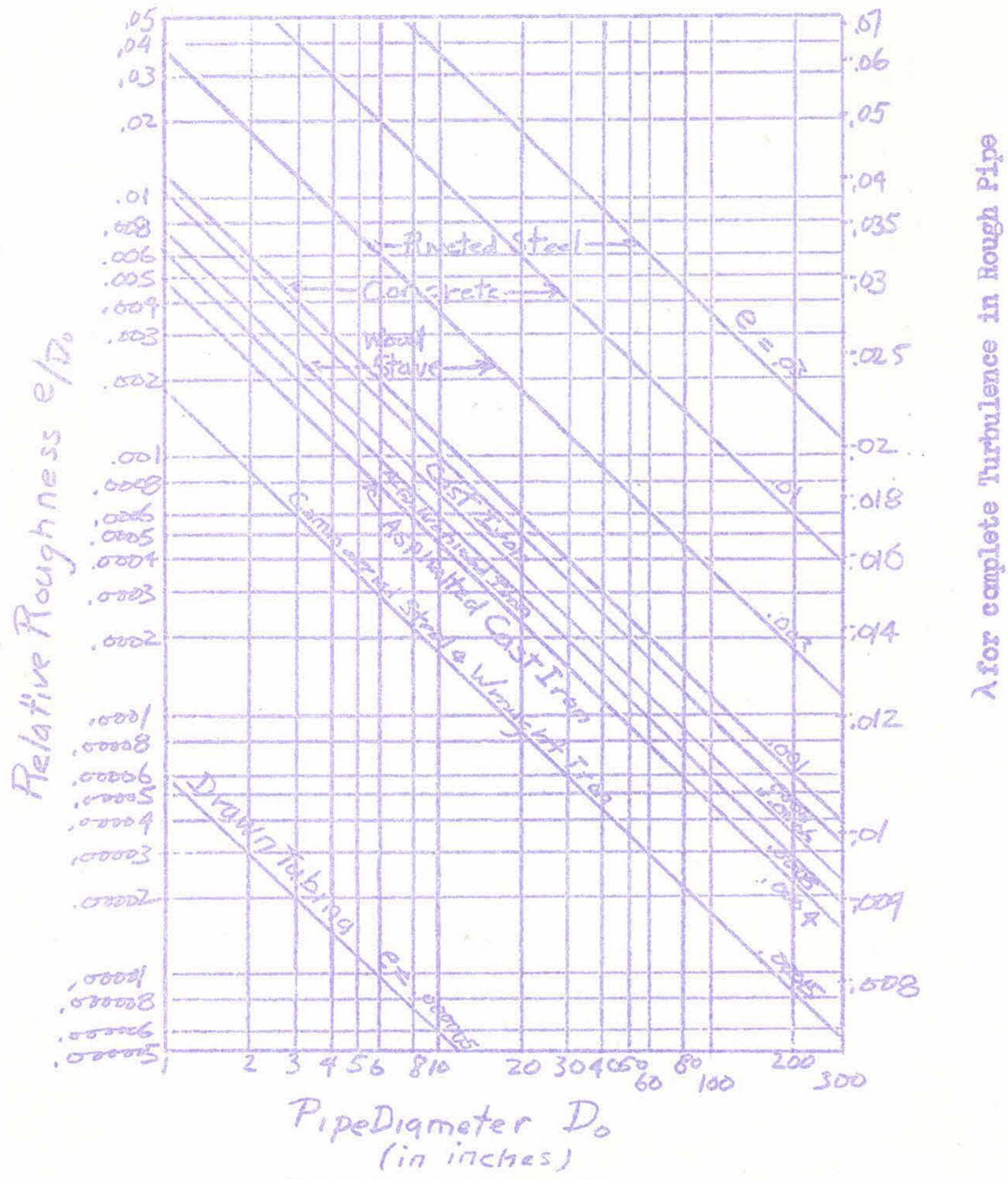
Fig 1-10



Friction Factors for Clean Commercial Pipe

After Moody ('44)

Fig 1-11



Relative Roughness for

Clean Commercial Pipe

Fig 1-12

CHAPTER II

FUNDAMENTAL CONCEPTS OF TURBULENT FLOW

INTRODUCTION

In this chapter the treatment of idealized flow begun in Chapter I is continued, and the basic concepts of the mixing length theories of turbulent flow are introduced. A discussion is also given of certain important experimental techniques for determining fluid velocities.

CHAPTER II

FUNDAMENTAL CONCEPTS OF TURBULENT FLOW

2-1 Concepts of Flow

In order to emphasize the idea of time averages in fluid flow set forth in Equation (1-6) of the previous chapter, these ideas will be applied specifically to the time average of velocity in a fluid.

If an observer who is stationary with respect to the solid material which bounds the steady turbulent flow of a viscous fluid were able to measure the instantaneous velocity at a fixed point, it would be possible to resolve this vectorial quantity u_i into three mutually perpendicular components, u_{x_i}, u_{y_i}, and u_{z_i}. A steady state of flow may be characterized as follows: the time average of any component can be made to differ by an arbitrarily small amount from a constant by averaging over a long enough time interval. This idea can be expressed for the X component as follows,

$$\frac{1}{\theta}\int_{\theta_0-\theta/2}^{\theta_0+\theta/2} u_{x_i}\,d\theta = \bar{u}_x + \epsilon_x \tag{2-1}$$

with the condition that $\bar{u}_x$ is a constant and that ϵ_x can be made smaller than any preassigned quantity by taking θ sufficiently large. Similar equations may be written for the Y and Z components.

Any component of the instantaneous velocity, u_{x_i} for example, can be expressed as the sum of the average value $\bar{u}_x$ and of a superimposed turbulent or fluctuation component, u_{x_f} which has a time average equal to

ϵ_x[1] as follows:

$$u_{xi} = \bar{u}_x + u_{xf} \tag{2-2}$$

It is possible to characterize turbulent flow for non-steady-state conditions in the above manner, if the rate of change in the average value of each component with time is small compared to the rate of change in corresponding fluctuation velocity.

It is sometimes advantageous to describe averages in the above manner with respect to a coordinate at a given instant of time.

2-2 Transfer of Momentum

Consider two layers of fluid, A and B, of Figure 2-1. Each layer is bounded by two parallel planes which may be considered to extend indefinitely. (The planes which bound the fluid are all parallel to each other.) Consider the two layers to be moving, each as a solid body, with instantaneous velocities always parallel to the planes. Without further loss of generality, the coordinates may be taken as fixed on the layer A, the X-Z plane may be taken parallel to the bounding planes, and the X-axis may be taken parallel to the relative instantaneous velocity of one layer, B, with respect to the other layer, A. If the layer B has an instantaneous velocity u_{xi} with respect to the layer A, and if, at a given instant, in addition to the above described motion, a mass of fluid is being transfered parallel to the Y-axis uniformly from each surface element of layer A to layer B with a velocity u_{yi}, the mass transfer per unit area per unit time is ρu_{yi}[(2)]

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[1] The fact that the time average of u_{xf} is ϵ_x can be seen by substitution of Equation (2-2) into Equation (2-1).

(2) In this treatment the density of the fluid, ρ, is equal to the specific weight σ divided by the acceleration of gravity g.

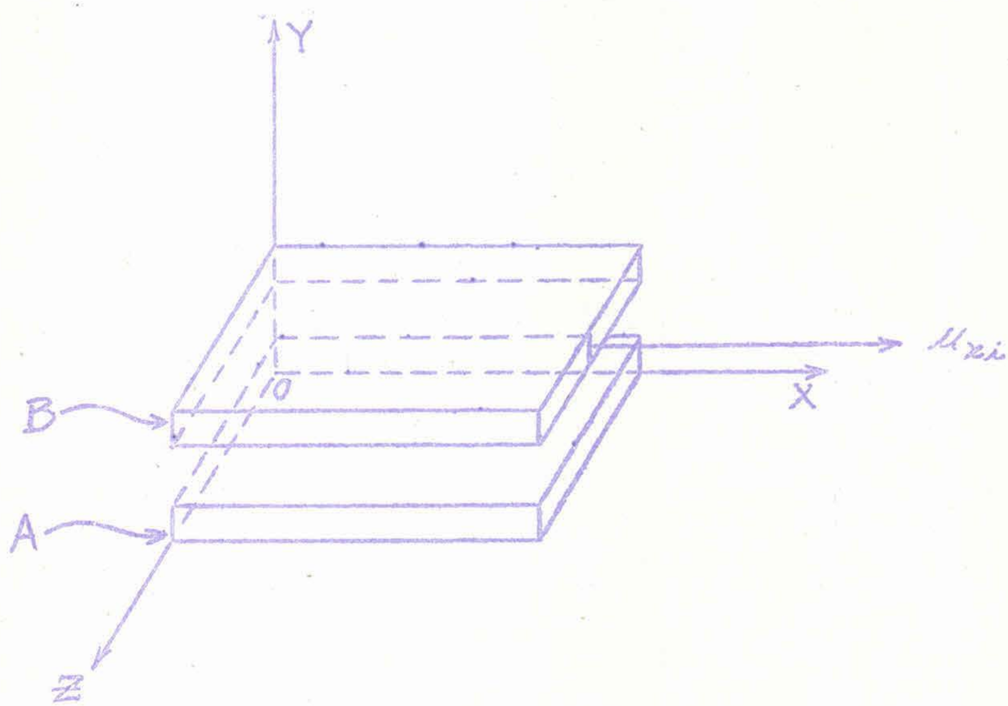

Fig. 2-1

and the time rate of exchange of momentum per unit area is $\rho u_{y_i} u_{x_i}$.
The time average of this quantity may be set equal to the average shearing stress, $\overline{\tau}_{zx}$. The stress $\overline{\tau}_{zx}$ acts on a plane perpendicular to the z-axis, and in a direction parallel to the X-axis. It acts to reduce the velocity of layer B relative to that of layer A.

The time average of the shearing stress may be written as,

$$\overline{\tau}_{zx} = \rho \overline{u_{y_i} u_{x_i}} \qquad [1] \qquad (2\text{-}3)$$

This result may be shown as follows: If each instantaneous value of velocity is expressed in terms of the sum of an average and a turbulent component, as in Equation (2-2),

$$\overline{\tau}_{zx} = \rho \overline{\left[\overline{u}_y + u_{y_f}\right] \left[\overline{u}_x + u_{x_f}\right]} \qquad [1] \qquad (2\text{-}4)$$

If steady-state conditions are assumed, the average values of u_{x_i} and u_{y_i} may be considered as constants, and

$$\overline{\tau}_{zx} = \rho\left[\overline{u}_y \overline{u}_x + \overline{u}_y \overline{u_{x_f}} + \overline{u}_x \overline{u_{y_f}} + \overline{u_{x_f} u_{y_f}}\right] \qquad [1] \qquad (2\text{-}5)$$

and since the average values of the turbulent components can be made arbitrarily small, as may be seen by Equation (2-1), in the limit,

$$\overline{\tau}_{zx} = \rho\ \overline{u}_y \overline{u}_x + \overline{u_{xf} u_{yf}} \qquad [1] \qquad (2\text{-}6)$$

But it was assumed originally that the planes were moving parallel to each other. Therefore, $u_y = 0$, and,

$$\bar{\tau}_{zx} = \rho \overline{u_{x_f} u_{x_f}} \qquad [1] \qquad (2\text{-}7)$$

2-3 Measurement of Velocity

For a detailed knowledge of the characteristics of a flow of a fluid, it is necessary to make measurements of velocity at various points in the flowing fluid.

Pilot tubes are ordinarily used to obtain average values of velocity. A description and list of references pertaining to this instrument has been given in a publication of the American Society of Mechanical Engineers.[1]

More information could be obtained about the mechanics of turbulent flow if it were possible to measure instantaneous values of velocity in turbulently flowing fluids. It is possible, by means of instruments which have been developed, to measure approximately the instantaneous values of velocity in different parts of turbulent flowing fluids. Thus the way is open to the determination of turbulent shear stresses as well as the establishment of correlations between events at various points in turbulently flowing fluids.

Hot wire anemometers, which usually consist of small heated wires placed with axis transverse to the direction of the stream, are sometimes employed to record the average value of the longitudinal speed[2] as well as

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1 Fluid Meters, Their Theory and Application, American Society of Mechanical Engineers, 29 W. 39th., New York City, Ed. 4, 1937, pages 38 and 130.
2 King, Engineering, 117, 136, 249, (1934).

the variation resulting from turbulence.[1]

In general, these instruments have been most useful in measurements involving average velocities of less than 100 ft/sec, since the sensitivity decreases at higher average velocities. The usual technique employed in the operation of hot-wire anemometers for the determination of average velocities consists of ascertaining the current necessary to maintain a constant average temperature of the wire as well as the resistance of the wire at the average velocity. The wire may be connected in one arm of a Wheatstone bridge circuit, and the current supplied to the bridge may be adjusted so that the bridge is balanced. The measurement of the current supplied and the resistance of the wire affords a measure of the velocity in the vicinity of the wire, which by suitable calibration can be interpreted in terms of the average velocity perpendicular to the wire.

It is possible to measure the turbulent fluctuations in velocity with a hot wire anemometer if the equipment is sufficiently refined. This measurement is usually accomplished by adding a vacuum tube amplifier to the detecting circuit of the Wheatstone bridge just described[2] and by decreasing the diameter of the hot wire. The heating current is adjusted in the usual manner until the bridge is balanced, and the small fluctuations in potential drop across the detecting circuit are amplified and observed on a cathode ray oscillograph or averaged by means of a suitable type of galvanometer. Such a circuit is indicated

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1 Dryden and Kuethe, NACA Rep. No. 320 (1929); Mock and Dryden, NACA Rep. No. 440 (1932)

2 Dryden and Kuethe, N.A.C.A. Rep. No. 320 (1929); Mock and Dryden, N.A.C.A. Rep. No. 448 (1932); Ziegler, Proc. Roy. Soc. Amsterdam, 34, 663, (1931).

schematically in Figure 2-2.

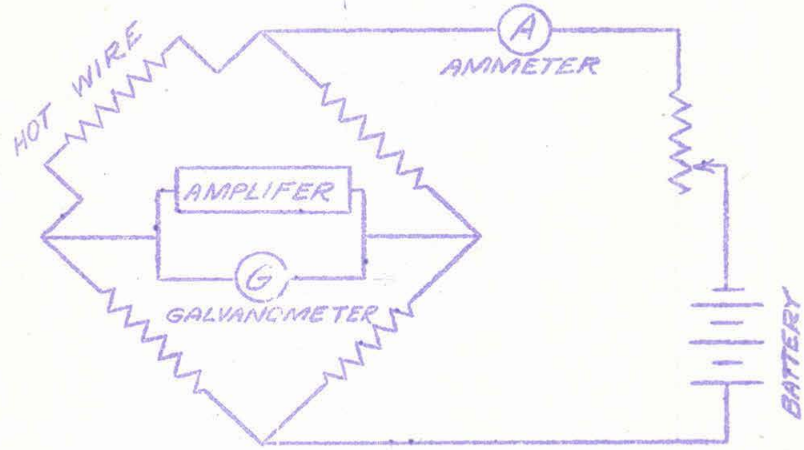

Fig. 2-2

2-4 Correlation

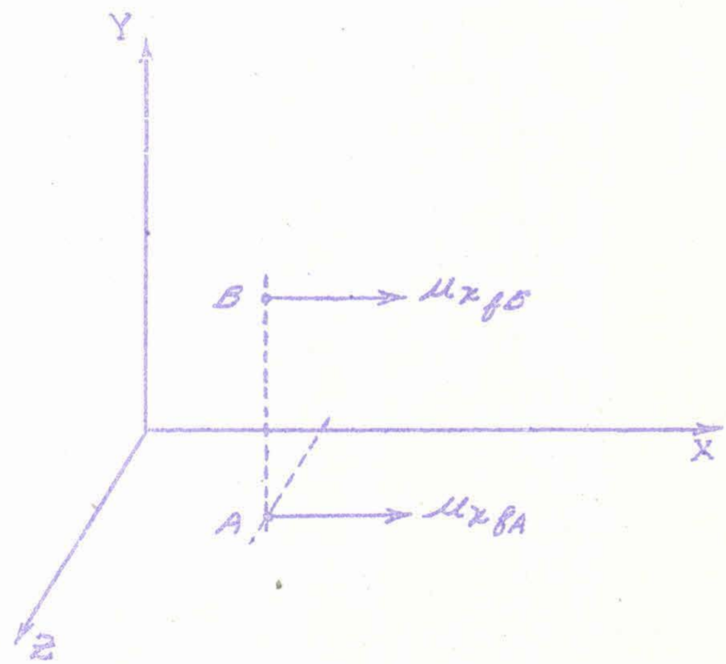

Fig. 2-3

It is possible to define a coefficient, R_y, which is a measure of the correlation between the turbulent fluctuations in the values of the velocity of a flowing fluid at two points in the flow. Thus if $u_{x_{fA}}$, at the point A, and $u_{x_{fA}}$, at the point B, which lies on a line drawn through A and parallel to the Y-axis, are the fluctuation values of the velocity of the fluid at A and B, where the X-axis is taken parallel to the mean flow, and the Y-axis is taken perpendicular to it, (see Figure 2-3) the correlation coefficient R_y, between these quantities, is defined as[1]

$$R_y = \frac{\overline{u_{x_{fA}} u_{x_{fB}}}}{\sqrt{\overline{u_{x_{fA}}^2}}\sqrt{\overline{u_{x_{fB}}^2}}} \qquad (2\text{-}8)$$

When the points A and B are close together a definite time correlation exists between the X components of the fluctuations of the velocity at each point, and the time average of their product is not zero.

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1 It is shown in most books on statistics[2] that if the simultaneous fluctuations, u_{xf_A} and u_{xf_B}, for example, are plotted on rectangular graph paper, that the straight line which best fits the points in the sense that the sum of the squares of the distances from the points to the line is a minimum, is

$$u_{xfA} = \frac{\sqrt{\overline{u_{xf_A}^2}}}{\sqrt{\overline{u_{xf_B}^2}}} R_y u_{xfB} \qquad (2\text{-}8a)$$

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2 Kenney Mathematics of Statistics D. Van Nostrand (1939); Rietz Mathematical Statistics Open Court Publ. Co. (1927).

It can be seen from Equation (2-8) that $R_y = 1$ when A and B are coincident. It has been established experimentally that, as $A \rightarrow B$, $R_y \rightarrow 1$. When points A and B are taken farther and farther apart, the time average of the product of the turbulent fluctuations of the velocity approaches zero, i.e., there is no correlation between the components. Hence when B recedes from A, $R_y \rightarrow 0$. The general variation of R_y with the distance apart, y, of A and B is indicated in Figure 2-4.

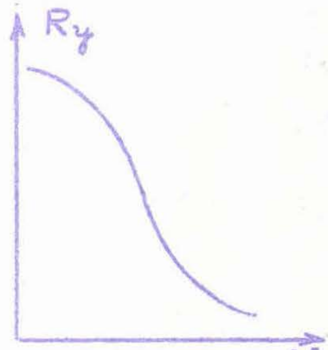

Fig. 2-4

Several electrical methods have been developed for the measurement of correlation coefficients. Prandtl and Reichardt,[1] for example, have applied the two amplified potential fluctuations obtained from separate bridge circuits connected to two hot wires to the vertical and horizontal plates of a cathode ray oscilloscope. Pictures having a long exposure time were obtained and analyzed for the degree of correlation between the fluctuation of the two velocities measured by the hot wires. A method described by Goldstein[2] involves the separate application of the two amplified potentials to the two coils of an alternating-current galvanometer.

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1 Prandtl and Reichardt, Deutsche Forshung, 21, 110 (1934).
2 Goldstein, Modern Developments in Fluid Dynamics, Oxford (1938) 1, 270.

2-5 Characteristic Length

A quantity having the dimensions of length may be defined from the correlation coefficient as follows:

$$l_y = \int_0^{Y_1} R_y d_y \qquad (2\text{-}9)$$

It is found experientally that the value of R_y is sensibly zero for all values of y above a certain value, which is designatedly by Y_1[1], so that the value of l_y becomes sensibly constant.

2-6 Mixing Length and Eddy Viscosity

It has been shown that the turbulent components of flow in two fluid layers parallel to the XZ plane, moving in the X-direction, give rise to a shear, the magnitude of which is,

$$\overline{\tau}_{zx} = \rho \overline{u_{x_f} u_{y_f}} \qquad [1] \qquad (2\text{-}7)$$

It would be advantageous, for purposes of practical application, if the quantity $\overline{u_{xf}u_{yf}}$ could be expressed in terms which are more easily measurable.

To this end, Prandtl postulated

$$\overline{u_{xf}u_{yf}} = l^2 \left(\frac{du_x}{dy}\right)^2 \qquad (2\text{-}10)$$

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1 Taylor, G, I., "Statistical Theory of Turbulence," Proc. Roy. Soc. London A151, 421 (1935).

where l is a quantity having the dimension of length, and which is characteristic of the turbulence existing at the point in question. The justification for the utilization this assumption may be taken from the results obtained by its use. If Equation (2-10) is assumed to be true, it is possible to calculate the Prandtl mixing length, l, from quantities which are easily measured.

Thus, if Equation (2-10) is substituted in Equation (2-7),[1]

$$\tau_{zx} = \rho l^2 \left(\frac{du_x}{dy}\right)^2 \qquad [1] \qquad (2\text{-}11)$$

Upon comparison with Equation (1-22) for the case of idealized flow between parallel plane plates which are a distance $2y_o$ apart, it is seen that

$$\tau_{zx_o} \frac{y}{y_o} = l^2 \left(\frac{du_x}{dy}\right)^2 \qquad [i,pp] \qquad (2\text{-}12)$$

This expression may be solved for l as follows:

$$l = \sqrt{\frac{\tau_{zx_o}}{\rho}} \sqrt{\frac{y}{y_o}} \frac{1}{\left(\frac{du_x}{dy}\right)} \qquad [i,pp] \qquad (2\text{-}13)$$

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1 Henceforth, except in special cases in which it is desired to emphasize the contrast between average and instantaneous values of shear stress, etc. the bar will be omitted.

The corresponding expression for idealized flow in a circular cylindrical conduit is, from Equation (1-12),

$$l = \sqrt{\frac{\tau_{rx_0}}{\rho}}\sqrt{\frac{r}{r_0}}\frac{1}{\left(\frac{du_x}{dr}\right)} \qquad [i,cc] \qquad (2\text{-}14)$$

It is to be emphasized that Equations (2-13) and (2-14) apply only to idealized flow, since Equations (1-22) and (1-12) do. ft^2/sec $lb\,sec^2/ft^4$

Equation (2-11) is sometimes written as,

$$\frac{\tau_{zx}}{\rho} = \left(l^2 \frac{du_x}{dy}\right)\frac{du_x}{dy} \qquad [l] \qquad (2\text{-}15)$$

or,

$$ft^2/sec \qquad \frac{\tau_{zx}}{\rho} = \epsilon\frac{du_x}{dy} \qquad [l] \qquad (2\text{-}16)$$

The quantity ϵ, usually called the eddy viscosity, is analogous to the quantity $\nu = \frac{\eta}{\rho}$, the kinematic viscosity.[1] The eddy viscosity can be thought of as the ratio of a quantity which is a measure of the shearing stress between two layers of fluid moving parallel to each other and the density. The shearing stress results from the exchange of momentum between the layers which occurs by turbulent fluctuations in velocity. The molecular

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1 The definition of eddy viscosity used in this work follows the practice of Von Kármán[1] and differs from that used by Bakhmeteff[2] in that the latter defines the eddy viscosity as $\rho l^2 \frac{du_x}{dy}$.

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1 Von Kármán, T., Trans. Am. Soc. Mech. Eng., 61, 705, (1939).
2 Bakhmeteff, The Mechanics of Turbulent Flow, Princeton Univ. Press (1941).

kinematic viscosity can be thought of as an analogous quantity which is the ratio of the stress resulting from the exchange of momentum which occurs by molecular agitation between the layers and the density.

2-7 Dimensionless Equations

In general it is of value to obtain equations which are independent of the absolute values of the various physical quantities which may change from one application to another. It can be seen by inspection of Equation (2-14) that a dimensionless form may be written as follows:

$$\frac{1 \dfrac{du_x}{dr}}{\sqrt{\dfrac{\tau_{rx_0}}{\rho}}} = \sqrt{\frac{r}{r_0}} \qquad [i, cc] \qquad (2\text{-}17)$$

The quantity $\sqrt{\frac{\tau_{rx_0}}{\rho}}$ may be seen to have the dimensions of a velocity, since the left side of Equation (2-17) must be dimensionless in order to be in agreement, dimensionally, with the right side. This fact may also be shown, of course, by the substitution of the fundamental units for τ_{rx_0} and ρ.

It is logical to designate the quantity $\sqrt{\frac{\tau_{rx_0}}{\rho}}$ as a velocity, which is characterized by the shearing stress at the wall and by the density of the fluid,

$$u_* = \sqrt{\frac{\tau_{rx_0}}{\rho}} \qquad [i, cc] \qquad (2\text{-}18)$$

If the quantity, u_*, usually called the "friction velocity", is substituted into Equation (2-17), it may be seen that,

$$\frac{1 \dfrac{du_x}{dr}}{u_*} = \sqrt{\frac{r}{r_0}} \qquad [i, cc] \qquad (2\text{-}19)$$

$$\frac{1\,\frac{du_x}{dy}}{u_*} = \sqrt{1 - \frac{y_d}{y_o}} \qquad [i, pp] \qquad (2\text{-}25)$$

Two dimensionless parameters which will be used later are defined:

$$r_d^+ = \frac{u_* r_d}{\nu} \qquad [i, cc] \qquad (2\text{-}26)$$

$$u_x^+ = \frac{u_x}{u_*} \qquad [i, cc] \qquad (2\text{-}27)$$

It is easy to see that r_d^+ is dimensionless, since the right side of Equation (2-26) has the dimensions of a Reynolds number. The quantity u_x^+ may be seen to be dimensionless since it is the ratio of two quantities each of which has the dimensions of velocity.

The Equations corresponding to (2-26) and (2-27) for two dimensional flow are, respectively,

$$y_d^+ = \frac{u_* y_d}{\nu} \qquad [i, pp] \qquad (2\text{-}28)$$

$$u_x^+ = \frac{u_x}{u_*} \qquad [i, pp] \qquad (2\text{-}29)$$

2-8 The Laminar Film

If flow in a conduit having solid walls is considered, it is evident that the fluctuation component of the velocity normal to the solid wall can be made arbitrarily small by choosing a point close enough to the wall. This idea implies that the shearing stress due to turbulence can be made is small as any preassigned quantity, by considering a region close enough to the wall. This fact may be seen by a consideration of Equation (2-7).

By a consideration of Equation (1-12) it is seen that $\tau = \tau_0 \frac{r}{r_0}$, for idealized flow. Hence, it is implied that the shear stress near the wall, in a turbulently flowing fluid must be transmitted through the layers of fluid near the wall by molecular transfer of momentum. In order for the high shear stresses encountered in turbulently flowing fluids to be transmitted by molecular transfer of momentum, very high velocity gradients must be present in the fluid near the wall. This fact can be seen from consideration of Equation (1-2). The average axial velocity in a conduit containing turbulently flowing fluid would then be zero at the wall, rise to a very high value in a short distance out from the wall, and then would rise more gradually at greater distances from the wall.

A graphical representation of such a velocity distribution is given in Figure 2-5.

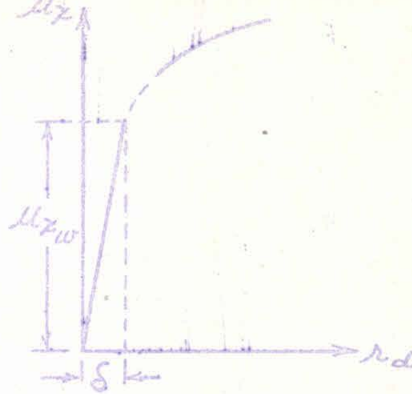

Fig. 2-5

The thickness of the laminar film is represented as δ. The velocity gradient across the film is $\frac{u_{xw}}{\delta}$. Hence by Equation (1-2),

$$\frac{\tau_{rx_0}}{\eta} = \frac{u_{xw}}{\delta} \qquad [i,cc] \qquad (2\text{-}30)$$

Since $\frac{\tau_{rx_0}}{\rho} = u_*^2$ by Equation (2-18), and since $\nu = \eta/\rho$ by definition,

$$\frac{u_{xw}}{\delta} = \frac{u_*}{\nu} \qquad [i,cc] \qquad (2\text{-}31)$$

so

$$\delta = \frac{\nu}{u_*} \, \frac{u_{xw}}{u_*} \qquad [i,cc] \qquad (2\text{-}32)$$

Equation (2-31) may also be written in another way, if the velocity gradient in the film is taken as u_x/r_d, where $u_x \leq u_{xw}$ and $r_d < \delta$:

$$\frac{u_x}{r_d} = \frac{u_*^2}{\nu} \qquad [i,cc] \qquad (2\text{-}33)$$

Or,

$$\frac{u_x}{u_*} = \frac{u_* r_d}{\nu} \qquad [i, cc] \qquad (2\text{-}34)$$

It is seen at once, by Equations (2-26) and (2-27) that Equation (2-34) may be rewritten as,

$$u_x^+ = r_d^+ \qquad [i, cc] \qquad (2\text{-}35)$$

It is to be remembered that the relation (2-33) is true only in the laminar film.

The expression for flow between parallel plates, corresponding to Equation (2-35) is

$$u_x^+ = y_d^+ \qquad [i, pp] \qquad (2\text{-}36)$$

It is now possible to define the condition of the surface of the inside of a conduit for a given flow as "rough", in case the maximum height of the protuberances e is greater than the thickness of the boundary layer δ, and as "smooth" in case the maximum height is less than that thickness. This idea is portrayed graphically in Figure 2-6.

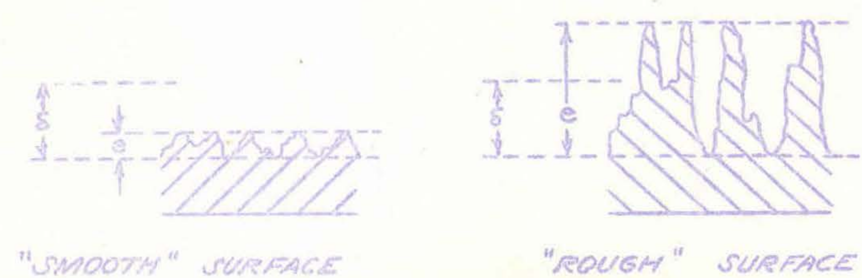

Fig. 2-6

CHAPTER III

FUNDAMENTAL EQUATIONS OF TURBULENT FLOW

INTRODUCTION

In this chapter a number of velocity distributions are derived for idealized flow in circular pipes and between parallel plates from the von Kármán similarity theory, the Prandtl mixing length hypothesis, and the Taylor vorticity transport theory, and the results compared with the existing experimental data.

Since none of these theories is admittedly more than a first approximation, the agreement between these approximate theories and experiment is only fair to good.

CHAPTER III

FUNDAMENTAL EQUATIONS OF TURBULENT FLOW

There are several methods available which permit approximate calculation of average velocity distributions in turbulent flow. The methods are based upon various hypotheses which have been advanced for prediction of values of the mixing length[1]. Three of these hypotheses will be considered in the present chapter.

3-1 The Similarity Hypothesis

The similarity hypothesis proposed by von Kármán[2] can be completely expressed only in mathematical terms. Goldstein[3] presents a somewhat simplified version of the theory and summarizes the results which are of interest. These are, for the case of idealized turbulent flow between parallel flat plates:

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1 It was pointed out in Chapter II that there is as yet no generally accepted definition of mixing length. This situation arises from the fact that mixing length theories cannot be subjected to complete experimental verification. The best that can be done is to compare the values of ℓ calculated on the basis of the theories with those expected from a priori considerations. It is, of course, possible to make assumptions regarding the variation of ℓ with respect to velocity distribution or position relative to flow boundaries, etc., and then compare calculated with observed velocity distributions.

2 Von Kármán, Th., Mechanische Ähnlichkeit und Turbulenz, Nach. Ges. Wiss., Göttingen (1930); Proc. 3rd. Internat. Congress for Applied Mechanics, Stockholm, (1930), 1, pp. 85-93.

3 Goldstein, S, Modern Devlopments in Fluid Dynamics (Oxford) 1938 pp. 347-352.

$$l = k \frac{\frac{du}{dy_d}}{\frac{d^2u}{dy_d^2}} \qquad [i, pp] \quad (3\text{-}1)$$

$$\tau_{zx} = \rho l^2 \left(\frac{du}{dy_d}\right)^2 \qquad [i, pp] \quad (3\text{-}2)$$

$$-\frac{1}{\rho}\frac{dP}{dx} = l^2 \frac{du}{dy_d} \cdot \frac{d^2u}{dy_d^2} \qquad [i, pp] \quad (3\text{-}3)$$

where y_d is the perpendicular distance from the nearest wall, x is the distance in the direction of flow and u is the average point velocity in the x- direction. For idealized turbulent flow in circular conduits, the corresponding results are:

$$l = \frac{k \frac{du}{dr}}{\frac{d^2u}{dr^2} - \frac{1}{r}\cdot\frac{du}{dr}} \qquad [i, cc] \quad (3\text{-}4)$$

$$\tau_{zx} = \rho l^2 \left(\frac{du}{dr}\right)^2 \qquad [i, cc] \quad (3\text{-}5)$$

$$\frac{d(r\tau_{zx})}{r\,dr} = \rho l^2 \frac{du}{dr}\left(\frac{d^2u}{dr^2} - \frac{1}{r}\cdot\frac{du}{dr}\right) \qquad [i, cc] \quad (3\text{-}6)$$

where r is the distance from the center of the conduit and u is the average point velocity in the x- direction.

It should be noted that (3-1) can be obtained simply from dimensional arguments without recourse to the full mathematical treatment of it may be assumed that:

1. The value of l at any point in the turbulent stream depends only upon the distribution of average velocity in the vicinity of the point.

2. The derivatives of velocity with respect to position in the stream of order higher than the second may be neglected in their effect upon the mixing length.

A frame of reference moving with the average velocity of the point under consideration may be taken, and as a consequence the mixing length will not involve u directly. Utilizing the Pi theorem[1] the equation :

$$\phi\left(l, \frac{du}{dy_d}, \frac{d^2u}{dy_d^2}\right) = 0 \qquad [i, PP] \quad (3\text{-}7)$$

may be written from which is obtained the dimensionless grouping

$$l \frac{\frac{d^2u}{dy_d^2}}{\frac{du}{dy_d}} = \phi'(c) = \text{constant} \quad [i, PP] \quad (3\text{-}8)$$

or

$$l = k \frac{\frac{du}{dy_d}}{\frac{d^2u}{dy_d^2}} \qquad [i, PP] \quad (3\text{-}1)$$

Elaborations of the theory[2] given above show that in addition l may depend upon $\frac{\frac{d^2u}{dy_d^2}}{\frac{d^3u}{dy_d^3}}$, $\frac{\frac{d^3u}{dy_d^3}}{\frac{d^4u}{dy_d^4}}$ etc. The simpler result of (3-1) can be valid only if all the quotients are proportional, a circumstance which may be shown[3] to occur only for flow in the absence of a pressure gradient.

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1 Buckingham, E., Model Experiments and the Forms of Empirical Equations, Trans. ASME, vol. 37 (1915)

2 Goldstein, Pg. 351

3 Goldstein, Pg. 351

It should be noted that (3-1) will cease to be of value where either $\frac{du}{dy_d}$ or $\frac{d^2u}{dy_d^2}$ is equal to zero, and that the formula will break down in the center of the channel.

3-2 Idealized Turbulent Flow Between Parallel Plates

Equation (1-12) can be rewritten for two-dimensional flow between parallel flat plates as

$$\tau_{zx} = \tau_0 \left(1 - \frac{y_d}{y_0}\right) \qquad [i, pp] \quad (3-9)$$

combining (3-1), (3-2), (2-18) and (3-9) the following equation results[1]

$$\frac{\frac{d^2u}{dy_d^2}}{\left(\frac{du}{dy_d}\right)^2} = -\frac{k}{u_*} \frac{1}{\sqrt{1 - \frac{y_d}{y_0}}} \qquad [i, pp] \quad (3-10)$$

This equation may be integrated directly to yield

$$\frac{1}{\frac{du}{dy_d}} = -2k \frac{y_0}{u_*} \sqrt{1 - \frac{y_d}{y_0}} + C_1 \quad [i, pp] (3-11)$$

In order to evaluate the constant which appears in the equation, it is assumed that for sufficiently high Reynolds numbers, the viscous layer is very thin and $\frac{du}{dy_d}$ is very large near the wall. The exact value of $\frac{du}{dy_d}$ will depend upon the thickness of the layer, but as a first approximation,

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1 The minus sign is chosen for the radical since $\frac{d^2u}{dy_d^2}$ will be negative over the region of integration.

it is taken to be infinite. Utilizing the boundary conditions $\frac{du}{dy_d} = \infty$ at $y_d = 0$ the constant is found to be

$$C_1 = 2k\left(\frac{y_0}{u_*}\right) \qquad [i, pp] \quad (3\text{-}12)$$

When the constant is incorporated in (3-11) and the equation is rearranged, the following results

$$\frac{du}{dy_d} = \frac{u_*}{2ky_0} \cdot \frac{1}{1-\sqrt{1-\frac{y_d}{y_0}}} \qquad [i, pp] \quad (3\text{-}13)$$

Using the substitution $1-\frac{y_d}{y_0} = x^2$, the equation becomes

$$u = -\frac{u_*}{k}\int \frac{x\,dx}{1-x} + C_2 \qquad [i, pp] \quad (3\text{-}14)$$

Performing the indicated integration and substituting the original variable in the integrand, the following results:

$$u = \frac{u_*}{k}\left[\sqrt{1-\frac{y_d}{y_0}} + \ln\left(1-\sqrt{1-\frac{y_d}{y_0}}\right)\right] + C_2 \qquad (3\text{-}15)\ [i, pp]$$

Noting that $u = u_m$ at $y_d = y_0$, it is apparent that $C_2 = u_m$ or

$$\frac{u_m - u}{u_*} = -\frac{1}{k}\left[\sqrt{1-\frac{y_d}{y_0}} + \ln\left(1-\sqrt{1-\frac{y_d}{y_0}}\right)\right] \qquad (3\text{-}16)\ [i, pp]$$

This is a dimensionless expression for the velocity deficiency as a function of position in the channel and may be used to check the validity of the similarity hypothesis by comparison with experimental data. Figure 3-1 presents such comparison based upon the data of Dönch[1] and Nikuradse[2]. In conformity with the procedures adopted by Goldstein[3], a small additive constant has been incorporated in (3-16) in the following manner

$$\frac{u_m - u}{u_*} = -\frac{1}{k}\left[\sqrt{1-\frac{y_d}{y_o}} + \ln\left(1-\sqrt{1-\frac{y_d}{y_o}}\right)\right] + b \qquad \begin{matrix}(3\text{-}17)\\ [i, pp]\end{matrix}$$

to permit a better fitting of the data at values of 0.3 and 0.7 for $1 - \frac{y_d}{y_o}$. The values of the constants required for best fit are $k = 0.245$ and $b = -0.172$, both dimensionless. It will be noted that b is quite small in comparison with the scale of Figure 3-1 and produces only a small vertical (downward) displacement of the theoretical curve.

The variation of the von Kármán mixing length with position in the channel is presented in terms of the dimensionless ratio $\frac{l}{y_o}$ in Figure 3-2. This curve is based upon the equation

$$\frac{l}{y_o} = 2k\left(\sqrt{1-\frac{y_d}{y_o}}\right)\left(1-\sqrt{1-\frac{y_d}{y_o}}\right) \qquad [i, pp]\ (3\text{-}18)$$

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1 Dönch, Forschungsarbeiten des Ver. deutsch Ing. No. 282 (1926).

2 Nikuradse, Forschungsarbeiten des Ver. deutsch. Ing. No. 289 (1929).

3 Goldstein, Pg. 352.

which may be obtained either by differentiating (3-16) twice and substituting in (3-1) or by combining (3-10), (3-13) and (3-1). It should be noted that the von Kármán mixing length is zero at $y = 0$ and $y_d = y_o$.

A second dimensionless expression for velocity deficiency as a function of position in the channel may be obtained from Equations (3-1) and (3-3). By equating the forces acting on an element of volume between parallel plates, as in Equation (1-21),

$$-\frac{dP}{dx} = \frac{\tau_{xo}}{y_o} \qquad [i, pp] \quad (3-19)$$

or

$$-\frac{1}{\rho}\frac{dP}{dx} = \frac{1}{y_o}\cdot\frac{\tau_{xo}}{\rho} = \frac{1}{y_o}u_*^2 \qquad [i, pp] \; (3-20)$$

since $u_*^2 = \frac{\tau_{xo}}{\rho}$ by definition.

By combining (3-20) with (3-3) as an intermediate step

$$\frac{u_*^2}{y_o} = l^2\frac{dU}{dy_d}\cdot\frac{d^2U}{dy_d^2} \qquad [i, pp] \; (3-21)$$

and then combining this result with (3-1), the following is obtained:

$$-\frac{\frac{d^2u}{dy_d^2}}{\left(\frac{du}{dy_d}\right)^3} = \frac{y_o k_1^2}{u_*} \quad {}^{(1)} \quad [i,pp]\ (3\text{-}22)$$

The minus sign in (3-22) arises from the fact that $\frac{du}{dy_d}$ is positive and $\frac{d^2u}{dy_d^2}$ negative over the range of integration. Equation (3-22) may be integrated directly to yield

$$\frac{1}{\left(\frac{du}{dy_d}\right)^2} = 2 y_o y_d \cdot \frac{k_1^2}{u_*^2} + C_1 \quad [i,pp]\ (3\text{-}23)$$

where $C_1 = 0$ since $\frac{du}{dy_d} \cong \infty$ at $y_d = 0$. Rearranging (3-23) after taking the square root of both sides, there results

$$\frac{du}{dy_d} = \frac{u_*}{k_1\sqrt{2y}} \cdot \frac{1}{\sqrt{y_d}} \quad [i,pp]\ (3\text{-}24)$$

which may be integrated directly to yield

$$u = \frac{\sqrt{2}\, u_*}{k_1\sqrt{y_o}} \cdot \sqrt{y_d} + C_2 \quad [i,pp]\ (3\text{-}25)$$

---

1 The subscript is applied to the constant k to distinguish it from the constant used in the equations derived by combination of (3-1) and (3-2). Although both constants originate in the k of (3-1), it has been found that somewhat different numerical values are required in order to fit the various analytical developments to experimental data.

where $C_2 = u_m - \frac{\sqrt{2}\,u_*}{k_1}$ since $u = u_m$ at $y_d = y_o$. This results in

$$\frac{u_m - u}{u_*} = \frac{\sqrt{2}}{k_1}\left(1 - \sqrt{\frac{y_d}{y_o}}\right) \qquad [i, pp]\,(3\text{-}26)$$

which is compared in Figure 3-3 with the same experimental data used in Figure 3-1. An additive constant has again been incorporated to permit better fit at values of 0.3 and 0.7 for $1 - \frac{y_d}{y_o}$. The values of the constants required for best fit are $k_1 = 0.165$ and $\beta = -0.736$. It will be noted that even utilizing this relatively large value of k_1, the agreement is not satisfactory for the region near the wall. Values of $\frac{\ell}{y_o}$ calculated from the expression

$$\frac{\ell}{y_o} = 2 k_1 y_d \qquad [i, pp]\,(3\text{-}27)$$

are shown in Figure 3-4. It is to be noted that in this case, the mixing length is zero at the wall and increases to a finite value $2 k_1 y_o$ at the center of the channel. A further point of interest is the fact that (3-26) gives a finite value of u at the walls whereas (3-16) gives an infinite negative value.

3-3 Idealized Turbulent Flow in a Circular Channel

Equations for flow in smooth circular conduits corresponding to (3-16) and (3-26) may be obtained by utilizing Equations (3-4), (3-5) and (3-6). By combining (3-4) and (3-5), there is obtained

$$\tau_{xy} = \rho \frac{k_2^2 \left(\frac{du}{dr}\right)^4}{\left(\frac{d^2u}{dr^2} - \frac{1}{r}\frac{du}{dr}\right)^2} \qquad [i, cc]\,(3\text{-}28)$$

which can be rearranged using the definition of u_* to obtain

$$\frac{r\cdot\frac{d^2u}{dr^2}-\frac{du}{dr}}{\left(\frac{du}{dr}\right)^2} = \frac{k_2}{u_*}\cdot\sqrt{r_o}\cdot\sqrt{r} \quad [i,cc] \quad (3\text{-}29)$$

which integrated directly to yield

$$-\frac{r}{\frac{du}{dr}} = \frac{2}{3}\cdot\frac{k_2}{u_*}\cdot r_o^{\frac{1}{2}}\cdot r^{3/2} + C_1 \quad [i,cc] \quad (3\text{-}30)$$

using the boundary conditions $\frac{du}{dr} \cong \infty$ at $r = r_o$, C_1 may be evaluated as

$$C_1 = -\frac{2}{3}\,\frac{k_2}{u_*}\,r_o^2 \quad [i,cc] \quad (3\text{-}31)$$

Then, combining (3-31) with (3-30) and rearranging, there results

$$-\frac{1}{r_o\frac{du}{dr}}\cdot\left(\frac{r}{r_o}\right) = \frac{2}{3}\cdot\frac{k_2}{u_*}\left[\left(\frac{r}{r_o}\right)^{3/2}-1\right] \quad [i,cc] \quad (3\text{-}32)$$

By introducing the variable $x = \frac{r}{r_o}$, the equation may be simplified somewhat and written in the integrated form

$$\frac{u_m - u}{u_*} = \frac{3}{2k_2}\int^{\frac{r}{r_o}}\frac{x\,dx}{x^{3/2}-1} \quad [i,cc] \quad (3\text{-}33)$$

The integral has been evaluated by Goldstein[1]. Figure 3-5 compares the theoretical results of Equation (3-33) with the experimental data of Stanton[2] and Nikuradse[3]. An additive constant has again been applied to permit better fit at $\frac{r}{r_o}$ equal to 0.3 and 0.7. The constants required for best fit are $k_2 = 0.171$ and $b = 0.420$. It is to be noted that the fit is poor near the wall. Figure 3-6 presents the von Kármán mixing length in terms of the dimensionless ratio $\frac{l}{r_o}$ based upon the equation

$$\frac{l}{r} = \frac{2}{3} k_2 \left(\sqrt{\frac{r_o}{r}} - \frac{r}{r_o} \right) \quad [i,cc] \quad (3\text{-}34)$$

obtained from (3-29), (3-32) and (3-4). In this case, it will be observed that l is zero at the wall and infinite at the center of the pipe. Furthermore the velocity at the wall based upon (3-33) is negative infinity.

A second relationship may be obtained for the velocity deficiency in circular conduits by combining Equations (3-4 and (3-6), to obtain

$$\frac{d}{r\,dr}\left(r \tau_{rx}\right) = \frac{2 \tau_{ox}}{r_o} = \frac{\rho k_3^2 \left(\frac{du}{dr}\right)^3}{\left(\frac{d^2u}{dr^2} - \frac{1}{r}\cdot\frac{du}{dr}\right)} \quad [i,cc] \quad (3\text{-}35)$$

Equation (3-35) may be rearranged to yield

$$\frac{2\frac{d^2u}{dr^2} - \frac{2}{r}\cdot\frac{du}{dr}}{\left(\frac{du}{dr}\right)^3} = -\frac{r_o k_3^2}{u_*^2} \quad [i,cc] \quad (3\text{-}36)$$

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[1] Goldstein, Pg. 354.

[2] Stanton, Proc. Roy. Soc., A **85** (1911) 366-375.

[3] Nikuradse, Ver. deutsch. Ing., Forschungsheft 356 (1932).

the negative sign being chosen to comply with the physical requirements. This equation has r^2 as an integrating factor. After multiplying through by r^2 and performing the integration there results

$$\frac{r^2}{\left(\frac{du}{dr}\right)^2} = -\frac{r_0 k_3^2}{u_*^2} \cdot \frac{r^3}{3} + C_1 \quad [i, cc] \quad (3\text{-}37)$$

Utilizing the boundary condition, $\frac{dU}{dr} \cong \infty$ at $r = r_0$, the constant may be evaluated, giving

$$C_1 = + \frac{r_0^4 k_3^2}{3 u_*^2} \quad [i, cc] \quad (3\text{-}38)$$

which, combined with (3-37) results in

$$\frac{r^2}{\left(\frac{du}{dr}\right)^2} = \frac{r_0 k_3^2}{3 u_*^2} \cdot (r_0^3 - r^3) \quad [i, cc] \quad (3\text{-}39)$$

Rearranging (3-39), there is obtained

$$\frac{1}{r^2\left(\frac{du}{dr}\right)^2} \cdot \left(\frac{r}{r_0}\right)^2 = \frac{k_3^2}{3 u_*^2} \left[1 - \left(\frac{r}{r_0}\right)^3\right] \quad [i, cc] \quad (3\text{-}40)$$

which after introduction of the variable $z = \frac{r}{r_0}$ becomes

$$\frac{z^2}{\left(\frac{du}{dz}\right)^2} = \frac{k_3^2}{3 u_*^2} (1 - z^3) \quad [i, cc] \quad (3\text{-}41)$$

By choosing the positive value of the radical after taking the square root, this equation may be integrated to yield

$$\frac{u_m - u}{u_*} = \frac{\sqrt{3}}{k_3}\int_0^{\frac{r}{r_o}} \frac{z\,dz}{(1-z^3)^{\frac{1}{2}}} \qquad [i,cc] \ (3\text{-}42)$$

This equation has been evaluated by Goldstein[1]. Figure (3-7) compares the theoretical results of (3-42) with some experimental data used in Figure 3-5. Again an additive constant has been used to permit better fit at values of $\frac{r}{r_o}$ equal to 0.3 and 0.7. The values of the constants for best fit are k_3 = 0.181 and b = 0.459. It is observed that the agreement with the experimental data is excellent. The von Kármán mixing length is presented as a function of the dimensionless ratio $\frac{l}{r}$ in Figure 3-8, the curve representing the function

$$\frac{l}{r} = \frac{2}{3} k_3 \left(\frac{r_o^2}{r^2} - \frac{r}{r_o} \right) \qquad [i,cc] \ (3\text{-}43)$$

obtained from (3-36), (3-39) and (3-4). It is noted that l is zero at the wall and infinite at the center of the conduit. Furthermore, the velocity is finite at the wall.

The similarity theory is the only theory so far proposed which yields a formula for l which may be used to evaluate velocity distribution in turbulent flow. All other mixture length theories require further assumptions regarding the variation of l with position or as a function of velocity distribution. The assumptions of the similarity theory

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[1] Goldstein, Pg. 355

are largely untested, and more over, even in the relatively simple cases to which the theory has been applied, there are regions in the field of flow where the assumptions break down. Goldstein[1] cites the following instances where these breakdowns are known to occur:

1. The restriction of the consideration of the turbulent mechanism at any point to the immediate neighborhood of the point requires that ℓ should be small compared with any typical linear dimension of the system under consideration, such for example as the half width of a channel or the radius of a pipe. (It should be noted, however, that this condition applies equally to all other currently advanced theories).

2. The assumption of similarity implies constant values of the ratios

$$u_{xf}^2 : u_{zf}^2 : u_{zf}^2 : \overline{u_{xf}u_{yf}} : \overline{u_{yf}u_{zf}} : \overline{u_{zf}u_{xf}}$$

where u_{xf}, u_{yf}, and u_{zf} are the turbulent fluctuation velocity components in the x, y, and z directions.

Experimental investigations[2] show that these conditions are violated in the center of the channel (where no correlation exists between the instantaneous velocity components) and near the walls where the viscous effects become important.

3-4 The Momentum Transfer Hypothesis

It was shown in Chapter II that the Reynolds stress may be expressed as

$$\overline{\tau}_{zx} = \rho\, \overline{u_{xf}u_{yf}} \qquad [1, pp] \quad (2-7)$$

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1 Goldstein, Pg. 350
2 Goldstein, Pg. 194

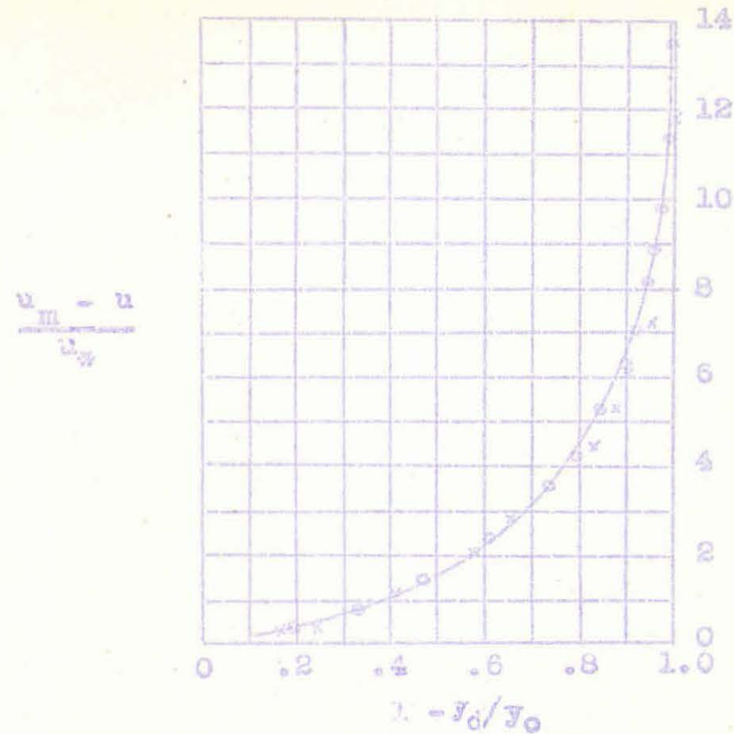

Figure
3-1

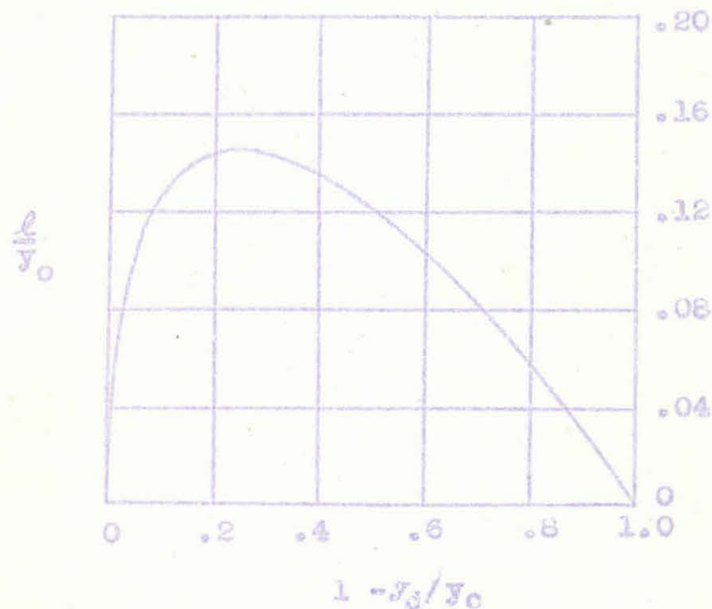

Figure
3-2

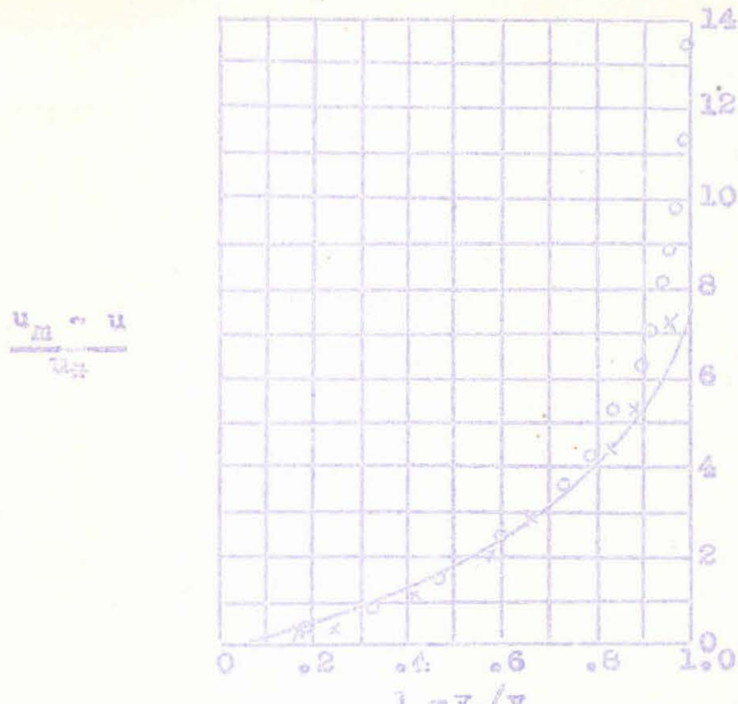

Figure
3-3

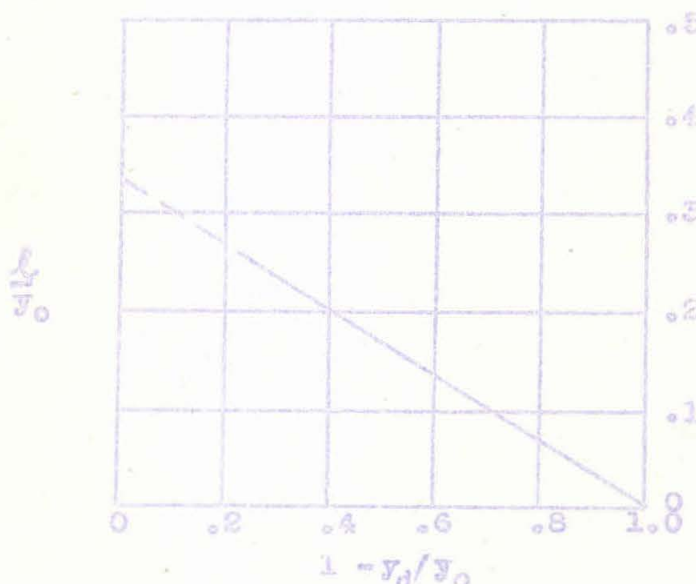

Figure
3-4

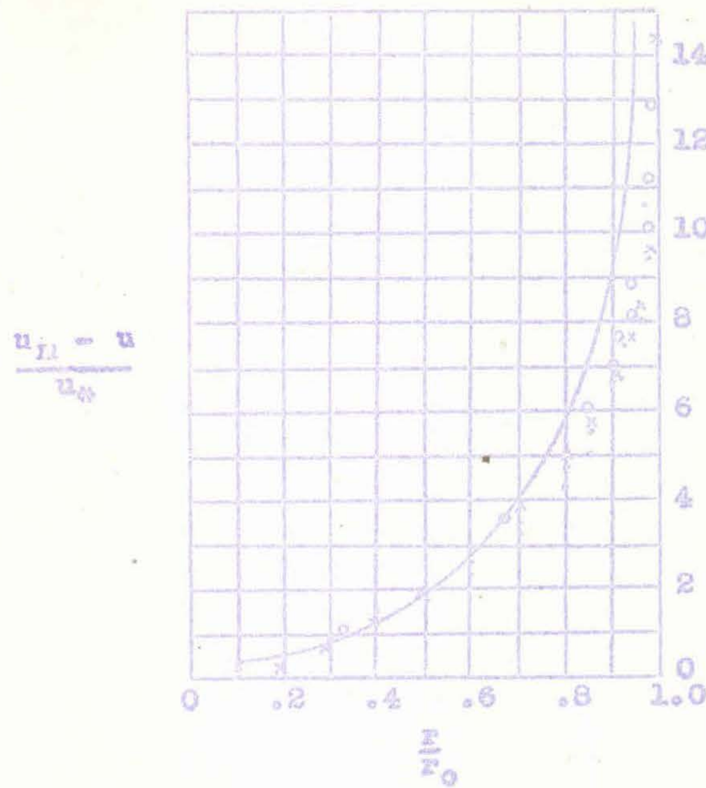

Figure
3-5

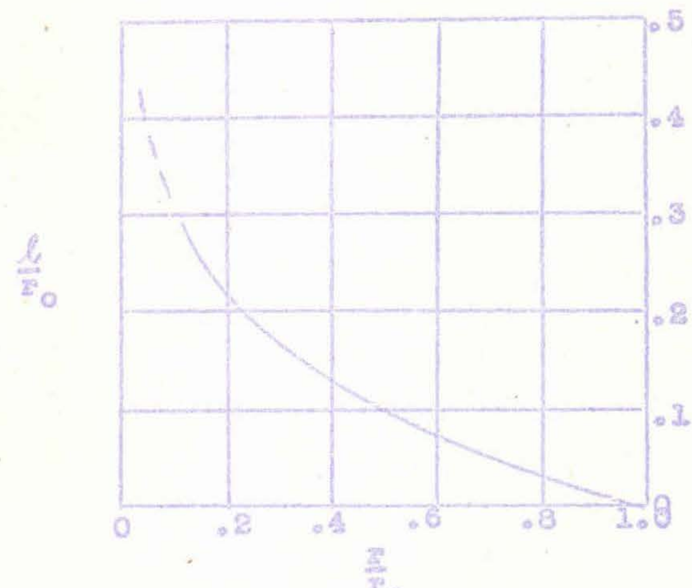

Figure
3-6

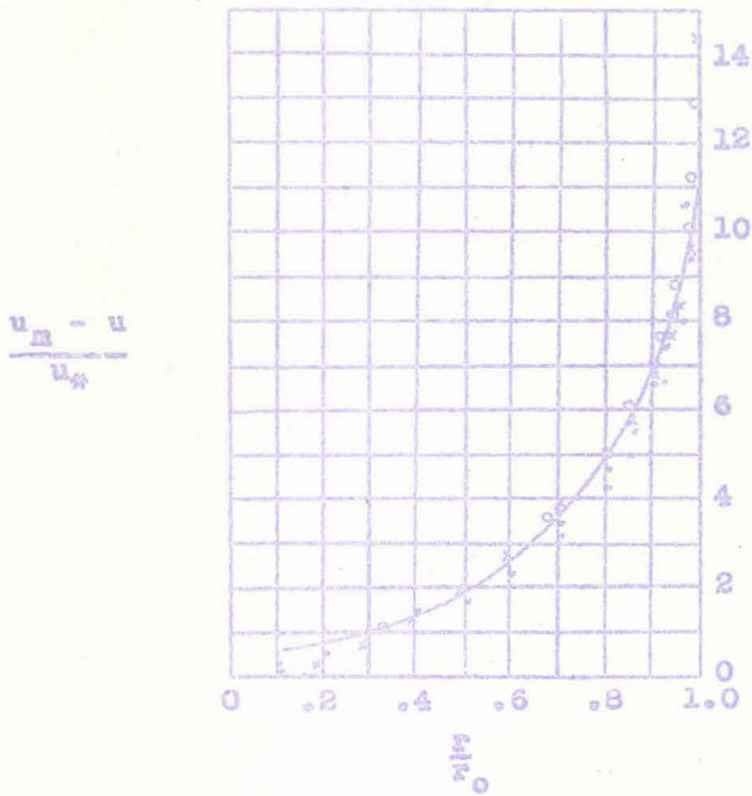

Figure
3-7

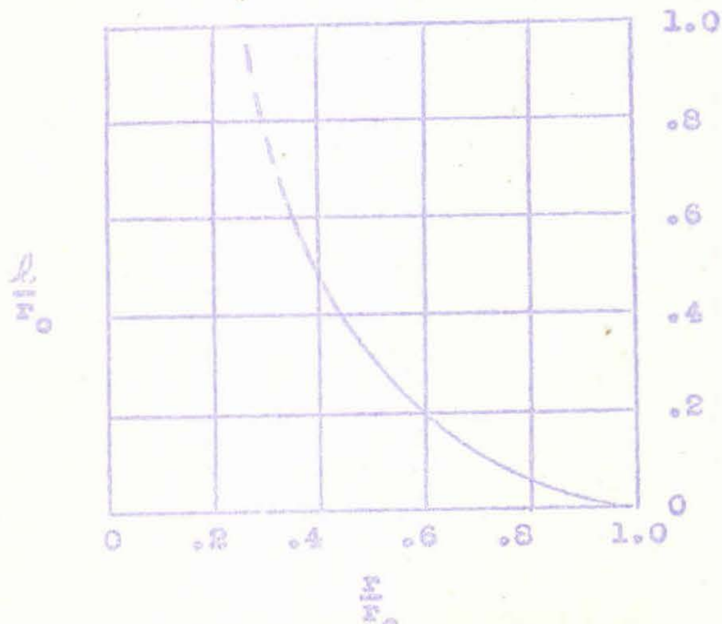

Figure
3-8

In order to utilize this expression in predicting the distribution of average velocity, it is necessary to determine how the expression $\rho \overline{u_{x6} u_{y6}}$ depends upon the average point velocity U and the boundary conditions. For this purpose Prandtl has proposed the following relationship:

$$\overline{u_{x6} u_{y6}} = - l^2 \frac{dU}{dy} \left| \frac{dU}{dy} \right| \quad ^{(1)} \quad [i, pp] \quad (3-44)$$

With this equation, the expression for the Reynolds stress becomes

$$\tau_{zx} = \rho l^2 \frac{dU}{dy} \left| \frac{dU}{dy} \right| \quad [i, pp] \quad (3-45$$

for idealized turbulent flow between flat, parallel plates and

$$\tau_{rx} = \rho l^2 \frac{dU}{dr} \left| \frac{dU}{dr} \right| \quad [i, cc] \quad (3-45A) .$$

for flow through circular conduits.

Before these equations may be applied to the solution of particular flow problems, it is necessary that an assumption be made regarding the variation of l with position in the channel. The experimental data of Nikuradse which were utilized in preparation of Figure 3-1 to 3-8 may also be used to compute values for the mixing length as a function of position in the channel. Figure 3-16 presents the results of such computation.

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1 For a discussion of the justification of such an hypothesis and the reason for the absolute value notation see Goldstein Pg. 206-208.

It will be observed that in the regionnear the wall, the mixing length is approximately a linear function of the distance from the wall. Therefore, as a first approximation, it is assumed that

$$l = k_4 y_d \qquad [i, pp] \quad (3-46)$$

throughout the channel. Justification of extension of linear variation to the remainder of the channel will depend upon the accuracy of the velocity distributions which are obtained.

3-5 Idealized Turbulent Flow between Flat, Parallel Plates

Equation (3-45) may be combined with (3-46) to eliminate l

$$\tau_{zx} = \rho k_4^2 y_d^2 \frac{du}{dy_d} \left| \frac{du}{dy_d} \right| \quad [i, pp] (3-47)$$

and (3-47) combined with (3-9) to obtain

$$\tau_{zx_0}\left(1 - \frac{y_d}{y_0}\right) = \rho k_4^2 y_d^2 \frac{du}{dy_d} \left| \frac{du}{dy_d} \right| \quad [i, pp] (3-48)$$

or

$$\tau_{zx_0} y = \rho k_4^2 y_0^3 \left(1 - \frac{y}{y_0}\right)^2 \left(\frac{du}{dy}\right)^2 \qquad (3-49) \; [i, pp]$$

where the positive value has been chosen for the velocity gradient term. The latter equation may be combined with the definition $u_* = \sqrt{\frac{\tau_0}{\rho}}$ and rearranged to yield

$$\frac{du}{dy} = \frac{u_* y^{\frac{1}{2}}}{k_4 y_0^{\frac{1}{2}} \left(1 - \frac{y}{y_0}\right)} \qquad [i, pp] \quad (3-50)$$

which may be integrated directly to the form

$$u = \frac{u_*}{k_4}\left\{ \ln\left[\frac{1+\left(\frac{y}{y_0}\right)^{\frac{1}{2}}}{1-\left(\frac{y}{y_0}\right)^{\frac{1}{2}}}\right] - 2\left(\frac{y}{y_0}\right)^{\frac{1}{2}} \right\} + C \quad [i, pp] \quad (3\text{-}51)$$

The constant of integration may be evaluated by setting $U = U_m$ at $y = 0$, or $C = U_m$. If this is incorporated in (3-51) and the resulting equation rearranged, there is obtained the dimensionless velocity deficiency equation

$$\frac{U_m - u}{u_*} = \frac{1}{k_4}\left\{ \ln\left[\frac{1+\left(\frac{y}{y_0}\right)^{\frac{1}{2}}}{1-\left(\frac{y}{y_0}\right)^{\frac{1}{2}}}\right] - 2\left(\frac{y}{y_0}\right)^{\frac{1}{2}} \right\} \quad [i, pp] \quad (3\text{-}52)$$

This equation satisfies the conditions $\frac{du}{dy} = 0$ at the center of the pipe, and $\frac{du}{dy} = \infty$ at the wall. The velocity at the wall is negative infinity. A comparison of this equation with the experimental measurements of Dönch and Nikuradse is shown in Figure 3-9, the value $k_4 = 0.23$ having been chosen to produce best fit at $\frac{y}{y_0} = 0.7$. It will be noted that the agreement is good except near the wall.

A similar derivation may be carried out for flow through circular conduits. The equations in this instance correspond directly to those for two dimensional flow with $y_d = r_d$ and $y = r$. The dimensionless velocity deficiency equation is then

$$\frac{U_m - u}{u_*} = \frac{1}{k_5}\left\{ \ln\left[\frac{1+\left(\frac{r}{r_0}\right)^{\frac{1}{2}}}{1-\left(\frac{r}{r_0}\right)^{\frac{1}{2}}}\right] - 2\left(\frac{r}{r_0}\right)^{\frac{1}{2}} \right\} \quad [i, pp] \quad (3\text{-}53)$$

This equation is compared with the experimental results of Stanton and

Nikuradse in Figure 3-10, a value of $k_s = 0.20$ having been chosen to give best fit at $\frac{r}{r_0} = 0.7$. The agreement is less satisfactory in this case, the theoretical curve diverging significantly from the experimental data near the wall.

It is interesting to note that Equation (3-52) may be presumed to apply to flow between both smooth and rough parallel walls, and correspondingly (3-53) to apply to flow through smooth or rough circular pipe under conditions in which the surface irregularities do not approach the magnitude of the wall separation or pipe diameter.[1] The experimental data presented in Figure 3-10, for example, are taken from work with smooth pipes, but if the comparison had been made with Nikuradse's data obtained with artifically roughened pipes,[2] similar agreement would have been obtained with $k = 0.24$.

The assumption that momentum is a transferable property involves the assumption that the fluctuating variations in pressure which exist in a turbulent field of flow are ineffective insofar as the mean transport of momentum is concerned.[3] The only case in which this can be proved to be true is when the momentum in the X-direction is transferred in the Y-Z plane by turbulent motion in which the streamlines initially parallel to the X-axis remain parallel to this axis throughout the motion. If the turbulent motion is two dimensional

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1 Goldstein, Pg. 344
2 Ver. deutsch. Ing., Forschung sheft 361 (1933)
3 Goldstein, Pg. 209

in the X-Y plane, it is shown in the preceding reference that vorticity[1] is a transferable property which fact may be used to develop theoretical equations for velocity deficiency in two-dimensional flow.

3-6 The Vorticity Transfer Hypothesis

The vorticity transfer hypothesis was initially proposed by G. I. Taylor is a result of his theoretical investigations of fluid flow.

- -

1 Vorticity is a vector quantity which is directly related to the angular velocity of a fluid element. It may be defined most precisely in mathematical terms (Page, Introduction to Theoretical Physics; Weatherburn, Advanced Vector Analysis). Its general meaning, however, may be explained (Stokes, Trans. Comb. Phil. Soc. 8 (1845) 309-310) by imagining an infinitesimally small sphere at any point of a fluid in motion to be suddenly solidified. If the resultant solid is found to have rotation, then the fluid possessed vorticity. Mathematically, the components of the vorticity in the X, Y, and Z directions are expressed as

$$\omega_x' = \frac{1}{2}\left(\frac{\partial u_{zi}}{\partial y} - \frac{\partial u_{yi}}{\partial z}\right)$$

$$\omega_z' = \frac{1}{2}\left(\frac{\partial u_{xi}}{\partial z} - \frac{\partial u_{zi}}{\partial x}\right)$$

$$\omega_y' = \frac{1}{2}\left(\frac{\partial u_{yi}}{\partial x} - \frac{\partial u_{xi}}{\partial y}\right)$$

If each component of the vorticity is zero, the fluid motion is termed irrotational. Other authors define the vorticity as twice this value (See Chapter V).

The mathematical statement of the generalized concept is quite involved and will not be reproduced here. However, for the case when the average velocity is in the X-direction and is a function of y only, the full equation of motion (see Chapter V), neglecting viscosity, may be written as

$$-\frac{\partial}{\partial x}\left(\frac{P}{\rho}+\frac{1}{2}(u_x+u_{xf})^2+\frac{1}{2}u_{yf}^2\right)=$$

$$\frac{\partial}{\partial t}(u_x+u_{xf})-u_{yf}\left(\zeta-\frac{du_x}{dy}\right) \qquad \binom{3-54}{[i,PP]}$$

where ζ is the turbulent vorticity. The mean value of this expression, since the average of $u_{xf}^2+u_{yf}^2$ will not vary with x, may be written as

$$\frac{1}{\rho}\frac{dP}{dx}=\overline{u_{yf}\zeta} \qquad [i,PP]\,(3\text{-}55)$$

from which Taylor postulated that the effect of turbulence is to communicate momentum at a rate of $\rho\overline{u_{yf}\zeta}$ to unit volume per unit time. Utilizing a modification of the Prandtl hypothesis (3-44), Taylor then set

$$\overline{u_{yf}\zeta}=l^2\left|\frac{du}{dy}\right|\frac{d^2u}{dy^2} \qquad [i,PP]\,(3\text{-}56)$$

from which there is obtained

$$\frac{1}{\rho}\frac{dP}{dx}=l^2\left|\frac{du}{dy}\right|\frac{d^2u}{dy^2} \qquad [i,PP]\,(3\text{-}57)$$

which is identical with (3-3) obtained by von Kármán on the basis of the similarity hypothesis.

Utilizing (3-46), (3-19) and the definition $u_* = \sqrt{\frac{\tau_0}{\rho}}$, (3-57) may be written as

$$u' u'' = \frac{u_*^2}{k_6^2 y_d^2 y_0} \qquad [i, pp] \quad (3\text{-}58)$$

or

$$u' u'' = \frac{u_*^2}{k_6^2 \left(1 - \frac{y}{y_0}\right)^2 y_0^3} \qquad [i, pp] \quad (3\text{-}59)$$

which may be integrated to yield

$$u'^2 = \frac{2 u_*}{k_6^2 y_0^2} \cdot \frac{1}{\left(1 - \frac{y}{y_0}\right)} + C \qquad [i, pp] \quad (3\text{-}60)$$

The constant may be evaluated by setting $u' = 0$ at $y = 0$ and combining the result in

$$u'^2 = \frac{2 u_*^2}{k_6 y_0^2} \left(\frac{\frac{y}{y_0}}{1 - \frac{y}{y_0}} \right) \qquad [i, pp] \quad (3\text{-}61)$$

the equation may be integrated to yield

$$u = \frac{\sqrt{2}\, u_*}{k_6} \left[\sin^{-1}\left(\frac{y}{y_0}\right)^{\frac{1}{2}} - \left(\frac{y}{y_0}\right)^{\frac{1}{2}} \sqrt{1 - \frac{y}{y_0}} \right] + C \quad [i, pp] \quad (3\text{-}62)$$

The constant may be calculated by setting $U = U_m$ at $y = 0$, which yields upon rearrangement the dimensionless velocity deficiency equation

$$\frac{u_m - u}{u_*} = \frac{\sqrt{2}}{k_6} \left[\sin^{-1}\left(\frac{y}{y_0}\right)^{\frac{1}{2}} - \left(\frac{y}{y_0}\right)^{\frac{1}{2}} \sqrt{1 - \frac{y}{y_0}} \right] \quad [i, pp] \quad (3\text{-}63)$$

This equation is compared with the experimental data of Dönch and Nikuradse in Figure 3-11. The constant k_6 is given the value 0.23 to obtain best fit at $\frac{y}{y_o} = 0.7$. This equation satisfies the condition $\frac{du}{dy}$ at $y = 0$ and $\frac{du}{dy} = \infty$ at $y = y_o$. The value of u is finite at the wall. Again the agreement is good in the center of the channel but poor near the wall.

For flow in circular conduits it is necessary to utilize the modified vorticity transfer hypothesis[1] in order to obtain the equation

$$\frac{1}{\rho}\frac{dP}{dx} = r^2 \left|\frac{du}{dr}\right| \left(\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr}\right) \quad [i,cc] \quad (3\text{-}65)$$

which may be combined with (3-46) and the definition of u_* to obtain

$$\frac{2\,u_*^2}{r_o} = k_7^2 (r_o - r)^2 \frac{du}{dr}\left(\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr}\right) [i,cc] \quad (3\text{-}66)$$

This may in turn be rearranged to yield

$$\frac{2\,u_*^2}{k_7 \left(1 - \frac{r}{r_o}\right)^2} = \frac{d^2u}{d\left(\frac{r}{r_o}\right)^2} \cdot \frac{du}{d\left(\frac{r}{r_o}\right)} + \frac{1}{\left(\frac{r}{r_o}\right)}\left(\frac{du}{d\left(\frac{r}{r_o}\right)}\right)^2 [i,cc] \quad (3\text{-}67)$$

which may be converted to an integrable form by multiplying both sides by $2\left(\frac{r}{r_o}\right)^2$

$$\frac{4\,u_*^2 \left(\frac{r}{r_o}\right)^2}{\left(1 - \frac{r}{r_o}\right)^2} = \frac{d}{d\left(\frac{r}{r_o}\right)}\left(\frac{r}{r_o}\,u'\right)^2 \quad [i,cc] \quad (3\text{-}68)$$

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1 Goldstein, Pgs. 213 - 214

Performing the integration, there is obtained

$$-4u_*^2\left[1-\frac{r}{r_0}-2\ln\left(1-\frac{r}{r_0}\right)-\frac{1}{\left(1-\frac{r}{r_0}\right)}\right]=\frac{r}{r_0}u'^2+C \qquad \begin{matrix}(3\text{-}69)\\ [i,cc]\end{matrix}$$

The constant may be evaluated by noting that $u'=0$ at $r=0$ taking the square root of both sides, there results

$$\frac{du}{d\left(\frac{r}{r_0}\right)}=-\frac{2u_*}{k_7}\left(\frac{r_0}{r}\right)\left[\frac{r}{r_0}-1+2\ln\left(1-\frac{r}{r_0}\right)+\frac{1}{\left(1-\frac{r}{r_0}\right)}\right]^{\frac{1}{2}} \qquad \begin{matrix}(3\text{-}70)\\ [i,cc]\end{matrix}$$

which may be integrated to yield

$$u=-\frac{2}{k_7}u_*\int\frac{r_0}{r}\left[\frac{r}{r_0}-1+2\ln\left(1-\frac{r}{r_0}\right)+\frac{1}{\left(1-\frac{r}{r_0}\right)}\right]^{\frac{1}{2}}d\left(\frac{r}{r_0}\right) \qquad \begin{matrix}(3\text{-}71)\\ [i,cc]\end{matrix}$$

The constant may be evaluated by setting $U=U_m$ at $r=0$. By combining the value with (3-69) there is often the dimensionless velocity deficiency equation

$$\frac{u_m-u}{u_*}=\frac{2}{k_7}\int_0^{\frac{r}{r_0}}\frac{r}{r_0}\left[\frac{r}{r_0}-1+2\ln\left(1-\frac{r}{r_0}\right)+\frac{1}{\left(1-\frac{r}{r_0}\right)}\right]^{\frac{1}{2}}d\left(\frac{r}{r_0}\right) \qquad \begin{matrix}(3\text{-}72)\\ [i,cc]\end{matrix}$$

This equation satisfies the condition $u'=0$ at $r=0$, and $u'=\infty$ at $r=r_0$. The velocity at the wall is finite. The integral has been evaluated by Taylor[1] and is plotted in Figure 3-12 in comparison with the experimental data of Stanton and Nikuradse. The agreement is observed to be quite good even in the

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1 Taylor, Proc. Roy. Soc. A 159 (1937) Pg. 503

vicinity of the wall.

3-7 Simplified Velocity Deficiency Equations

Somewhat simpler equations may be obtained for the velocity dificiency by considering the region near the wall in which the shear stress may be assumed constant with respect to radius, i.e. $\tau = \tau_0$. If this assumption is applied to the Kármán similarity hypothesis for two-dimensional flow, Equation (3-10) may be written as

$$\frac{\frac{d^2u}{dy_d^2}}{\left(\frac{du}{dy_d}\right)^2} = -\frac{k_8}{u_*} \qquad [i, pp] \quad (3\text{-}73)$$

which may be integrated to yield

$$\frac{1}{\left(\frac{du}{dy_d}\right)} = \frac{k_8}{u_*} y_d + C_1 \qquad [i, pp] \quad (3\text{-}74)$$

at the wall $\frac{du}{dy_d} = \infty$ and $y_d = 0$, or $C_1 = 0$. This results in

$$\frac{du}{dy_d} = \frac{u_*}{k_8}\frac{1}{y_d} \qquad [i, pp] \quad (3\text{-}75)$$

This equation may also be obtained from the Prandtl momentum transfer hypothesis by combining (3-45) (3-46) and the assumption that $\tau = \tau_0 =$ constant near the wall.

Equation (3-75) may be integrated to yield

$$u = \frac{u_*}{k_g} \ln y_d + C_2 \quad [i, pp] \; (3\text{-}76)$$

Subject to experimental investigation, Prandtl assumed that this equation might apply to the entire channel as well as to the region near the wall. This permitted him to evaluate the constant, for at $y_d = y_o$, $u = u_m$ or

$$\frac{u_m - u}{u_*} = \frac{1}{k_g} \ln \frac{y_o}{y_d} \quad [i, pp] \; (3\text{-}77)$$

Similarly for flow in circular conduits, there is obtained

$$\frac{u_m - u}{u_*} = \frac{1}{k_q} \ln \frac{r_o}{r_d} \quad [i, cc] \; (3\text{-}78)$$

since the Prandtl hypothesis applies to both two-dimensional flow between flat parallel plates and flow in circular conduits.

Comparisons of equations of the form of (3-77) and (3-78) with experimental data show that these equations comply quite well with experimental velocity deficiency data. Figure 3-15 is based upon Nikuradse's data previously used for Figures 3-5, 3-7, 3-10 and 3-12 and presents $\frac{u}{u_*}$ as a function of $\log \frac{r_d u_*}{\nu}$. Since ν is presumed constant, this is equivalent to (3-78). Similar agreement should be obtained for (3-77). In both cases it is found that a value of the constant approximately equal to 0.4 is required for best fit with the experimental data. When this value for k_g and k_q are substituted in (3-77) and (3-78) there are obtained the often used forms of a velocity deficiency equation

$$\frac{u_m - u}{u_*} = 2.5 \ln \frac{y_0}{y_d} \quad [i, pp] \quad (3-79)$$

or

$$\frac{u_m - u}{u_*} = 5.75 \log_{10} \frac{y_0}{y_d} \quad [i, pp] \quad (3-80)$$

and

$$\frac{u_m - u}{u_*} = 2.5 \ln \frac{r_0}{r_d} \quad [i, cc] \quad (3-81)$$

or

$$\frac{u_m - u}{u_*} = 5.75 \log_{10} \frac{r_0}{r_d} \quad [i, cc] \quad (3-82)$$

It should be pointed out that Equation (3-75), in common with the Kármán Equations (3-15) and (3-32), gives a finite value for the velocity gradient in the center of the channel. This is in variance with experimental observation. A further discrepancy arises in the fact that the mixing length calculated on the basis of the velocity gradient Equation (3-75) and in terms of the Prandtl mixing length

$$l = \frac{u_* \sqrt{1 - \frac{y_d}{y_0}}}{\left(\frac{du}{dy_d}\right)} \quad (3-83)$$

derived from 3-45) and (3-9) together with the definition of u_*, and by substituting (3-75) in (3-83) yields

$$\ell = k y_d \sqrt{1 - \frac{y_d}{y_0}} \qquad [i, pp] \; (3\text{-}84)$$

which is zero at the wall and the center of the channel, passing through a maximum at $y_d = \frac{2}{5} y_0$ as shown in Figure 3-14. Similar results are obtained for flow through circular conduits. Therefore, this curve may be compared with the variation of ℓ computed on the basis of the von Kármán similarly hypothesis in Figures 3-2, 3-4, 3-6, and 3-8.

It is of interest to note that certain investigators, notably Gebelein[1] have utilized the principles of statistical mechanics to investigate the general problems of fluid flow. In certain respects the approach appears to eliminate some of the inconsistencies noted in the preceding paragraphs. Although no attempt will be made to describe this method of analysis in this chapter, the theoretical curve obtained by Gebelein for mixing lengths is presented in Figure 3-15 along with the values calculated from Nikuradse's data of Figure 3-1. It is observed that the agreement is better in certain respects than that for the three hypothesis discussed in this chapter.

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1 Gebelein, Turbulenz Edwards Bros. (1944)

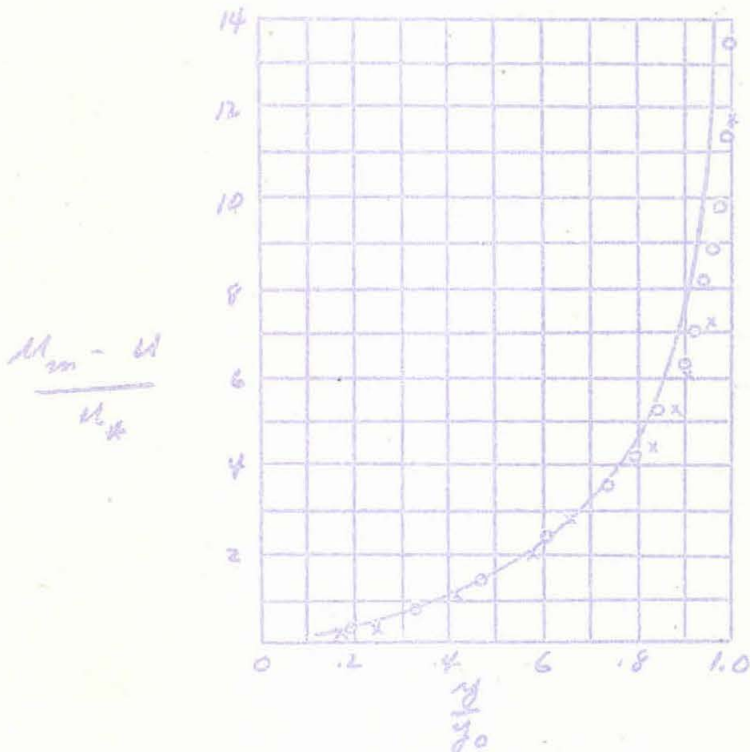


Figure 3-9

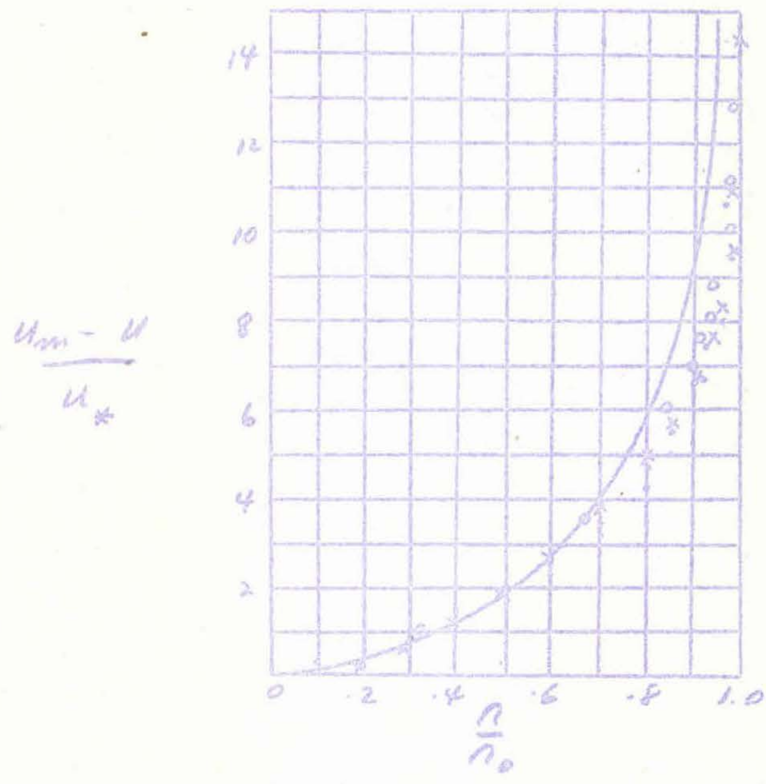


Figure 3-10

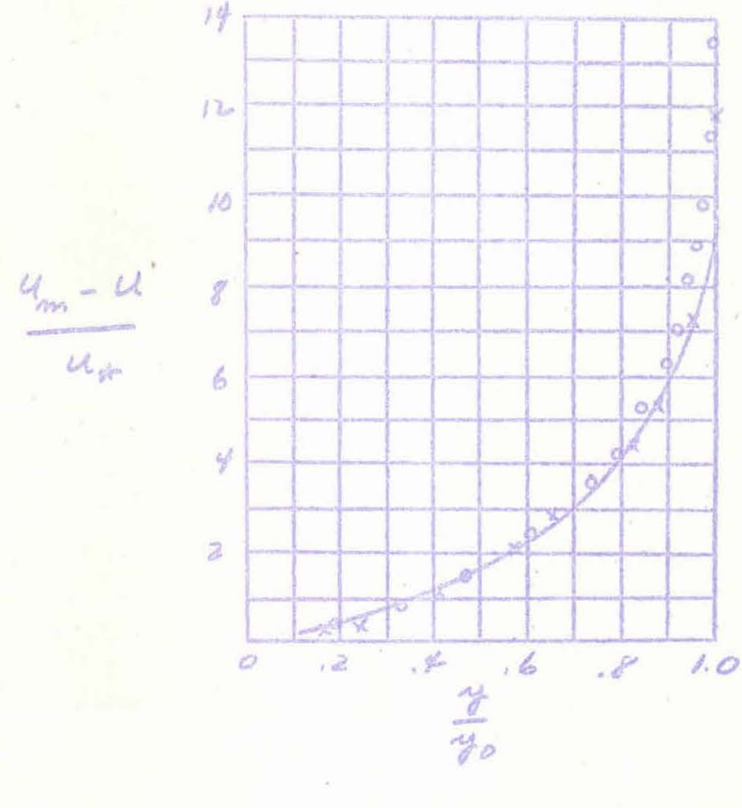


Figure 3-11

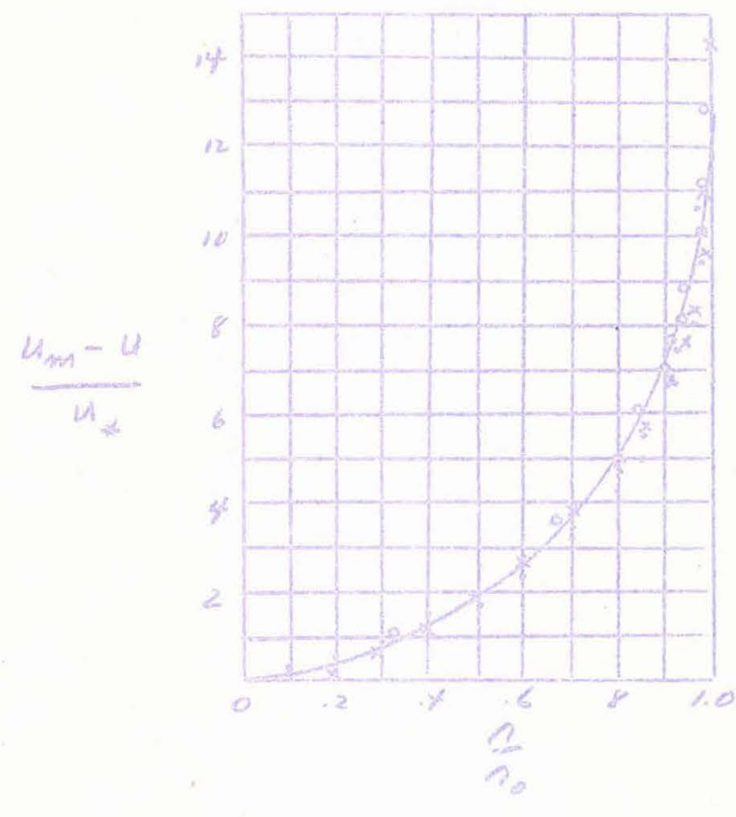


Figure 3-12

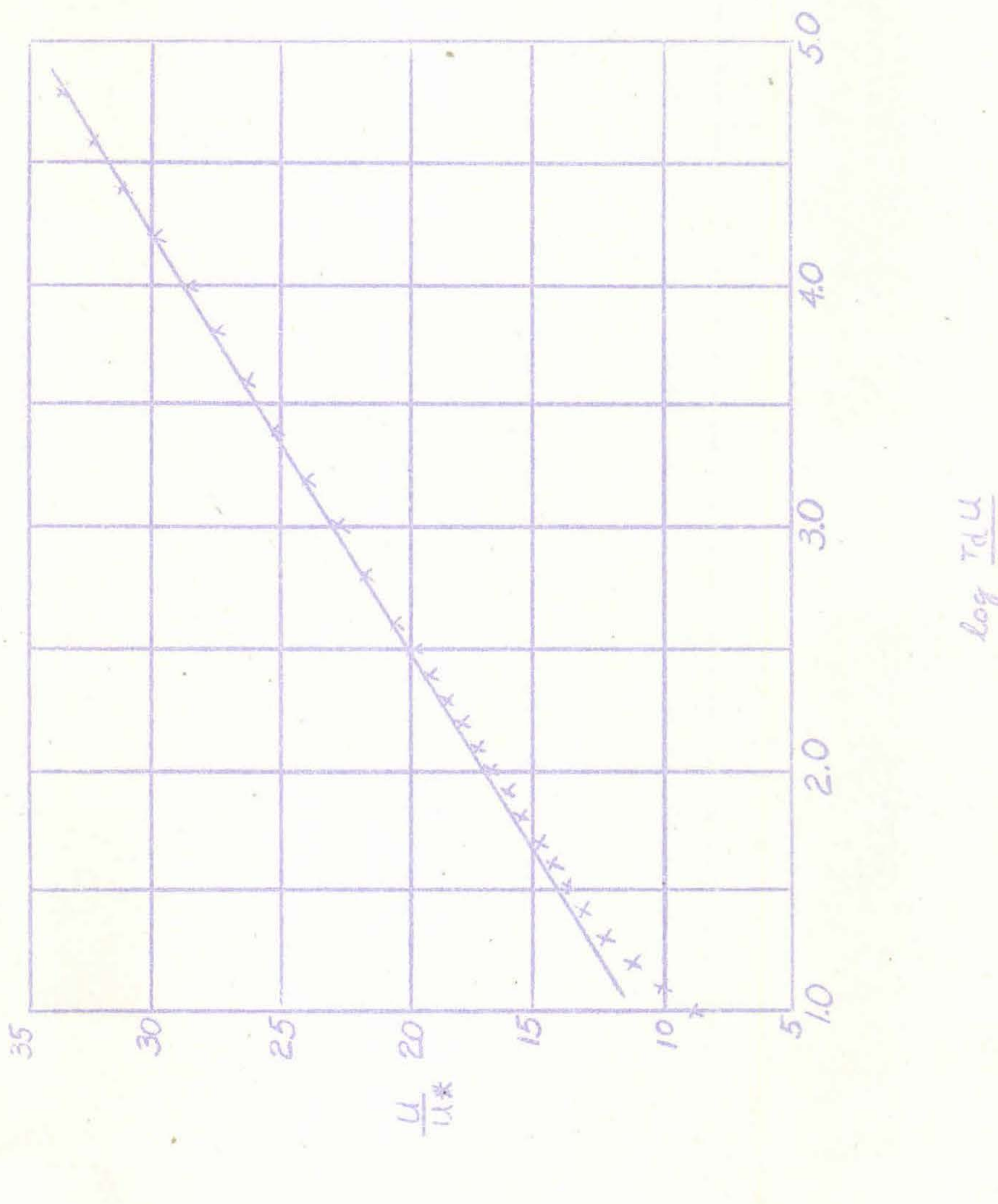
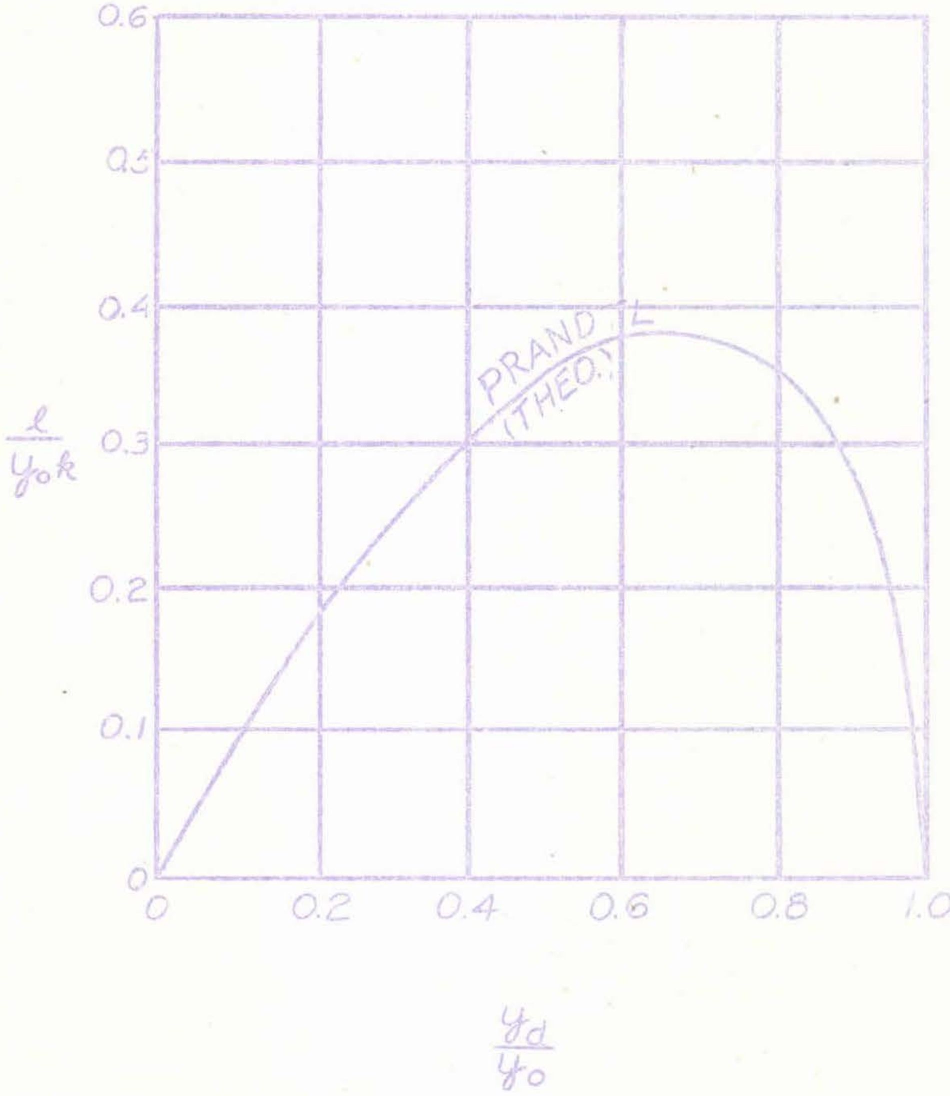
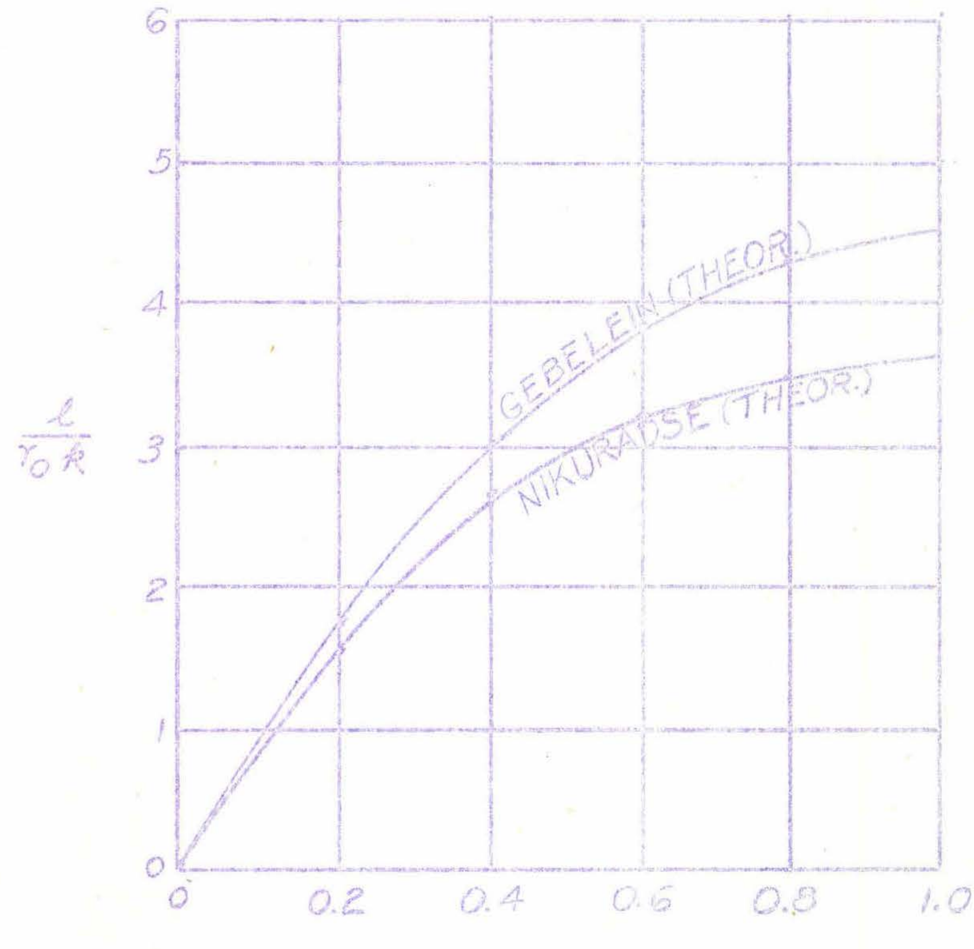


FIGURE 3-13



0.6
0.5
0.4
0.3
0.2
0.1
0
$\frac{l}{y_o k}$
0
0.2
0.4
0.6
0.8
1.0
PRANDTL
(THEO.)
$\frac{y_d}{y_o}$

FIGURE 3-14



6
5
4
3
2
1
0
0
0.2
0.4
0.6
0.8
1.0
GEBELEIN (THEOR.)
NIKURADSE (THEOR.)
$\frac{l}{r_0 k}$
$\frac{r}{r_0}$

FIGURE 3-15

CHAPTER IV

VELOCITY CURVES AND FRICTION FACTORS FOR TURBULENT FLOW

INTRODUCTION

Although none of the velocity distributions derived in Chapter III gives perfect agreement with experimental measurements, fortuitously the simplest of them (Eqs. (3-77) and (3-78)) is very accurate — as accurate as any of them are.

In this chapter this distribution is applied to the calculation of velocity distributions and friction factors in terms of easily calculable quantities both for smooth and rough pipe.

CHAPTER IV

VELOCITY CURVES AND FRICTION FACTORS FOR TURBULENT FLOW

4-1 Velocity Curves

After the theoretician has developed his equations, the matter of prime interest to the engineer is whether or not the equations may be presented in a generally applicable form yielding accurate answers to his problems. The only path open in the proof of the theoretical results is, of course, careful analysis of existing experimental data.

Of immediate concern here is the validity of the turbulent flow formulae developed with the aid of the length parameter l. The equations developed in Chapter III are mainly valid in the central portion of the flow stream and do not have satisfactory application near the conduit wall. Prandtl, however, suggested that the equation[1]

$$u_x = \frac{u_*}{k} \ln r_d + \text{constant} \quad [i,cc] \quad (4\text{-}1)$$

as developed on the basis of the momentum transfer theory[2] for flow near the walls of circular conduits be also applied to the central portion of the channel. The application has been made using (4-1) in the form

$$\frac{u_x}{u_*} = A + \frac{2.303}{k} \log \frac{r_d u_*}{\nu} \quad [i,cc] \quad (4\text{-}2)$$

- -

1 See Chapter III, Equation 3-76
2 See Chapter III also Goldstein, S., (ed) "Modern Developments in Fluid Dynamics," Oxford Press, (1938), II, 331-336

where A is a constant. Using 0.40 as the value of k[1] and 5.5 for the constant A, Equation (4-2) becomes

$$\frac{u_x}{u_*} = 5.5 + 5.75 \log \frac{r_d u_*}{\nu} \quad [i, cc] \quad (4-3)$$

and is found to give good agreement with Nikuradse's data[2] obtained for turbulent flow in smooth circular pipes. It is to be emphasized that 4-3 gives a generalized velocity distribution in close prediction of experiment for the region from the laminar layer to the center of the pipe. Considering the simplified character of the development of Equation (4-1), the agreement is fortunate.

To complete the analysis of the generalized velocity distribution, it is necessary to provide an equation describing the conditions from the wall to the edge of the laminar layer[3]. Such an equation is[4]

$$u^+ = r_d^+ \quad [i, cc] \quad (4-4)$$

which as noted was derivated in Chapter II.

Figure 4-1 shows the combination of (4-3) and (4-4) providing the generalized velocity distribution for turbulent flow in smooth circular pipes. Figure 4-2 gives a plot of u^+ versus r_d^+ in Cartesian coordinates showing that the generalized equations yield curves similar to actual

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1 Prandtl, L., Aerodynamic Theory, Durand (ed), 3 (Berlin, 1935) 140.
2 Nikuradse, J., Ver. deutsch. Ing., Forschungsheft 356, (1932).
3 Here, as in the remainder of the discussion, the laminar layer is taken as that region bounded by the channel wall and the envelope defined by the intersections of the extrapolated velocity curves in the turbulent and laminar sub-layer regions.
4 See Chapter II, Equation 2-30.

velocity traverses. Of interest in Figure 4-1 is the fact that the experimental data for $8 \leq y^{+} \leq 30$ are not particularly well expressed by the two-distribution laws. This latter region is the buffer layer which as noted earlier was not being considered in the analysis. It may be of further interest to provide for the treatment of the buffer layer.

In addition to the inadequacy of the generalized equations for the velocity distribution in the region of the buffer layer in turbulent flow, some investigators have noted that the equations also provide a discontinuity in the first derivative of the velocity curve with respect to the distance from the wall at the center line of the channel. The discontinuity has its origin in the fact that the equations were derived using the theory that the mixing length, ℓ, is zero at the center line. It has been observed[1] that the mixing length may instead have a maximum at the center line.

If it is desired, the velocity distribution equation for turbulent flow in smooth circular pipes may be applied to the specific problem of evaluating the average bulk velocity, U. It has been shown[2] that

$$\frac{U}{u_*} = \frac{u_{x\,max}}{u_*} - D \qquad [i,cc]\ (4\text{-}5)$$

D has been found to be approximately $\frac{3}{2}\frac{u_*}{k}$. Taking k as 0.4, D becomes 3.75. Since (4-3) gives

$$\frac{u_{x\,max}}{u_*} = 5.5 + 5.75 \log \frac{r_0 u_*}{\nu} \qquad [i,cc]\ (4\text{-}6)$$

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1 See Chapter III, Figure 3-16

2 Bakhmeteff, B., "The Mechanics of Turbulent Flow", Princeton University Press, (1936), 70

it is apparent that

$$\frac{U}{u_*} = 5.5 - 375 + 5.75 \log \frac{r_o u_*}{\nu}$$

or

$$\frac{U}{u_*} = 1.75 + 5.75 \log \frac{r_o u_*}{\nu} \qquad [i,cc] \quad (4\text{-}7)$$

Equation (4-1) through (4-7) apply specifically to flow in smooth circular pipes. An equation of the form (4-2) is also applicable to flow between infinite parallel planes of smooth surface, with r_o being replaced by the distance out from the wall y_d. See the results of Chapter VII for a comparison of the experimental and predicted valves of the velocity distribution for this case.

4-2 The Resistance Factor λ

The resistance factor λ for flow in smooth circular pipes may be obtained by considering the generalized form of (4-7)

$$\frac{U}{u_*} = A' + B \log \frac{r_o u_*}{\nu} \qquad [i,cc] \quad (4\text{-}8)$$

It has been shown in Chapter I by means of Equation (1-59) and (1-60) that

$$\frac{\tau_o}{\rho} = \frac{f}{2} U^2 = \frac{\lambda}{8} U^2 = u_*^2 \qquad [i] \quad (4\text{-}9)$$

By also noting that for circular pipes

$$R_e = \frac{2 r_0 U}{\nu} \qquad [cc] \quad (4\text{-}10)$$

Equation (4-8) gives

$$\frac{1}{\sqrt{\lambda}} = C' + B' \log (R_e \sqrt{\lambda}) \qquad [i, cc] \quad (4\text{-}11)$$

Equation (4-11) holds only for sufficiently large Reynolds's numbers such that a region of turbulence exists in the center of the pipe and extends to the end of the curve for the turbulent region as shown in Figure 4-1.

Data from Nikuradse[1] give a form of (4-10)[2] as

$$\frac{1}{\sqrt{\lambda}} = 2 \log (R_e \sqrt{\lambda}) - 0.8 \qquad [i, cc] \quad (4\text{-}12)$$

If, in Equation (4-8), the quantity r_0 were replaced by y_0 half the distance between two infinite parallel plates, an equation of the same type as (4-11) would be obtained. The constants of course should be obtained from experimental data collected for two-dimensional flow between smooth parallel planes.

4-3 Laminar Film Thickness

At the transition from the laminar to the turbulent region the friction distance parameter has been observed to have a constant value, N, analogous to the value of 2000 for Re in the transition from streamline to

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1 See Prandtl, L., "Neue Ergebnisse der Turbulenz Forschung," Z.d.V.D. I. No. 5. (1933).

2 See also Goldstein S., ed., "Modern Developments in Fluid Dynamics," Oxford Press (1938) II, 338.

turbulent flow. Thus the thickness of the laminar layer may be expressed in the equations:

$$y^*_\delta = \frac{u_* \delta}{\nu} = constant = N \qquad (4\text{-}13)$$

or

$$\delta = N \frac{\nu}{u_*} \qquad (4\text{-}14)$$

Since Equation (4-9) gives a relation between U and u_*, there is obtained from the latter equation and (4-14) the following expression:

$$\delta = \frac{N\nu}{U}\sqrt{\frac{8}{\lambda}} \qquad (4\text{-}15)$$

or noting that

$$Re = \frac{DU}{\nu}$$

one obtains

$$\frac{\delta}{D} = \frac{N\nu}{UD}\sqrt{\frac{8}{\lambda}} = \frac{N}{Re}\sqrt{\frac{8}{\lambda}} \quad (i,cc] \ (4\text{-}16)$$

As λ may be expressed as a function of Re in flow in smooth circular pipes (see Equation (4-11)), Equation (4-16) shows that the relative film thickness

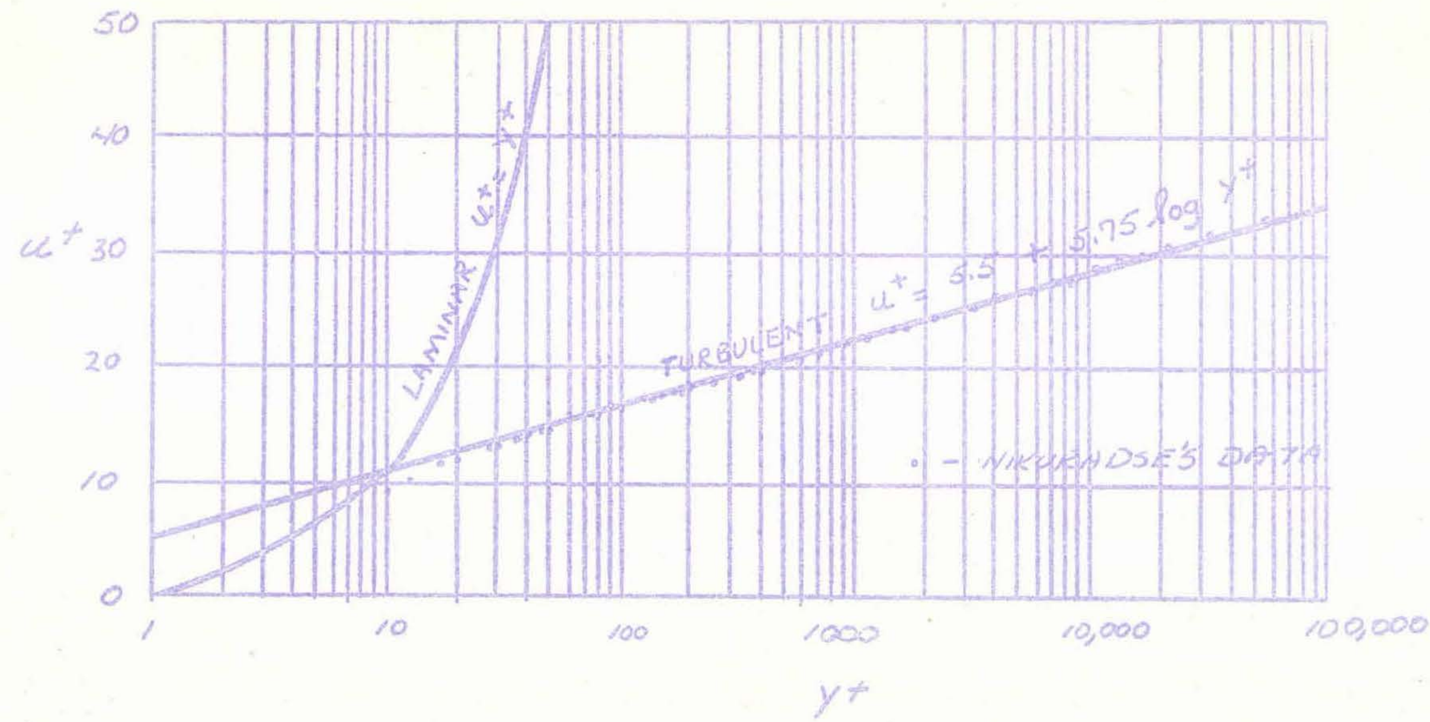

Figure 1.

Velocity Distribution Laws (Logarithmic Scale)

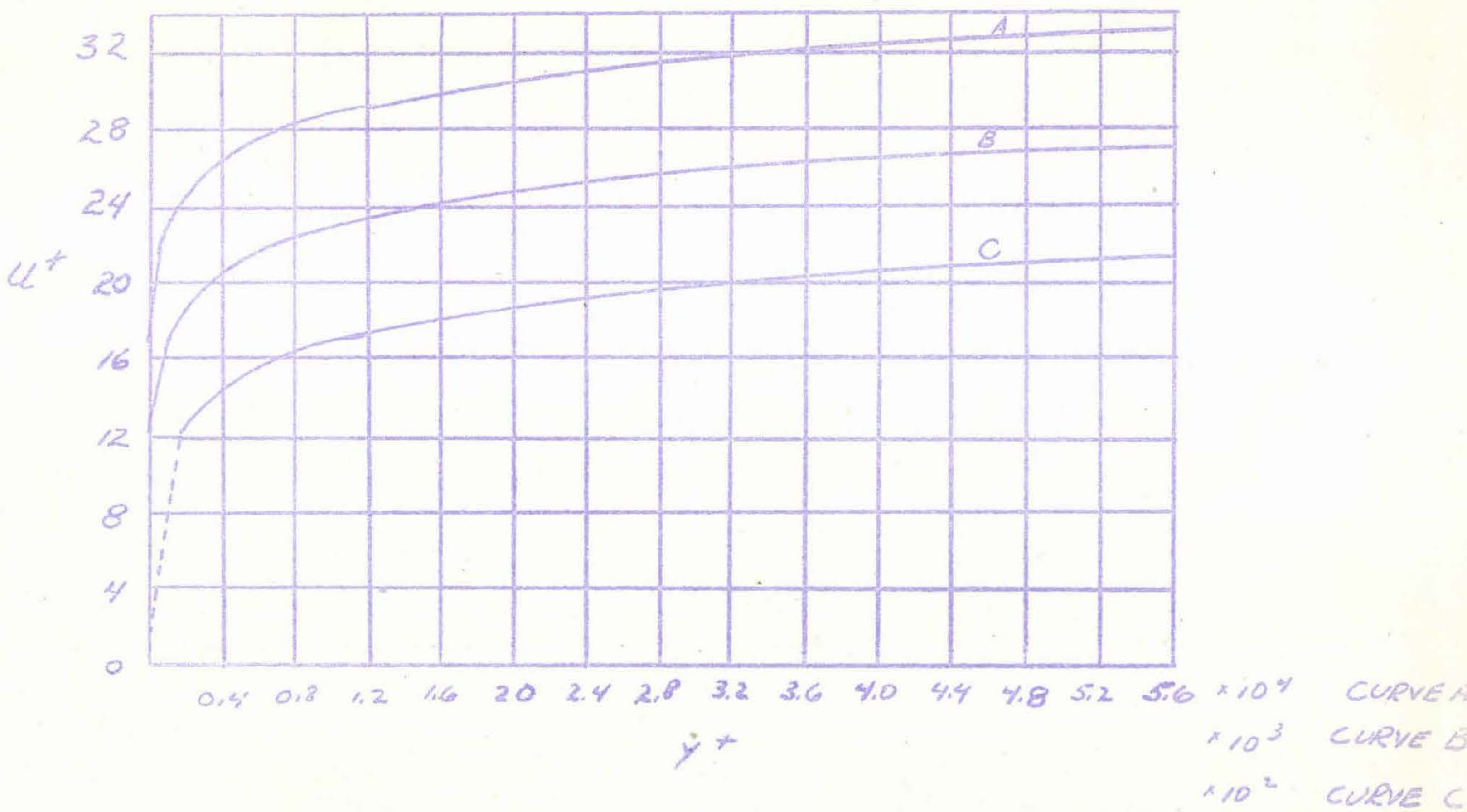

Figure 2.

Universal Velocity Distribution Curves

(Natural Coordinates)

is also a function only of Re. Similarly, replacing D, the pipe diameter, by $2y_o$, the distance between parallel planes, it is noted that (4-16) is also valid for flow between smooth parallel planes.

4-4 Velocity Curves

In smooth conduits, the Reynolds number is the independent quantity used to characterize the flow. When considering flow in rough conduits, however, a second parameter, the wall roughness, is also needed in describing the flow conditions.

The expressions developed in Chapter III are applicable to the determination of the velocity pattern in flow in both smooth- and moderately rough-walled[1] conduits. As noted in the present chapter, however, they do not provide for satisfactory treatment of velocities near the walls. Consequently, as in Part I, the wall equations are used and extrapolated in very satisfactory fashion through the central portions of the conduits being considered.

Before proceeding with the analysis it is first necessary to clarify how surface roughness is to be classified in uniting theory and fact. If only closely packed[2] surfaces are involved, then it is possible to describe the surface by noting the magnitude of the depth of roughness, e, perpendicular to the surface. The walls of circular conduits may be classified in the same roughness category by comparing the ratios of r_o/e. Similarly the surfaces of parallel plates are compared by noting the relative values of y_o/e.

- -

1 Here the smooth or moderately rough walled conduits are essentially those where the height, e, of a surface protroberance is less than the thickness of the laminar layer.
2 See Goldstein, S. (ed), Modern Developments in Fluid Dynamics, Oxford, 1938; p 376 Section 167 - Nikuradse used artifically roughened pipes where different sized sand grains allowed close packing and controlled values of r_o/e.

Of particular interest in the present discussion is the region of high Reynolds' numbers where the laminar films are relatively thin and λ becomes independent of Re. Here friction is proportional to the square of the velocity and wholly "rough" flow exists. The velocity distribution equations and those relating λ to r_o/e (or y_o/e) apply only to the "rough" flow. Nikuradse[1] showed that $eu_*/\nu \geq 100$ gives a hydraulically rough and $\frac{eu_*}{\nu} \leq 4$ a hydraulically smooth surface.

The equation

$$u_x = \frac{u_*}{k} \ln r_d + \text{constant} \qquad (4-1) \; [i, cc]$$

the form of which was suggested by Prandtl[2], is universal in nature and hence may be applied to flow in rough-walled pipes, rewriting in terms of the maximum velocity, taken at the center of the channel, and the velocity u_w at the inner edge of the laminar layer, there is obtained:

$$\frac{u_w}{u_*} = \frac{u_{max}}{u_*} - 5.75 \log \frac{r_o}{r_{dw}} \qquad [i, cc] \quad (4-17)$$

nothing that a value of $k = 0.4$ as used by Prandtl[3] is introduced. Since different wall conditions now exist because of the roughness of the surface, u_w/u_* is not constant as it was where smooth-walled conduits were being considered. Also $r_{d/w}$ may be expected to depend on the roughness, e, in the following manner:

$$r_{dw} = me \qquad (4-18)$$

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1 Nikuradse, J., Ver. Deutsch. Ing., Forschungsheft 361 (1933)
2 Prandtl, L., Zeitschr. des Vereines deutscher Ingenieure 77, 107-108, (1933).
3 Prandtl, L., Aerodynamic Theory (edited by Durand) 3, 140, (1935) Berlin.

where m depends on the shape and size of the wall protuberances. By substituting (4-18) in (4-17) and eliminating r_o and $u_{x_{max}}$ between the resulting equation and (4-16) there is obtained the expression

$$\frac{u_{max} - u_x}{u_*} = 5.75 \log \frac{r_o}{r_d} \qquad [i, cc] \; (4\text{-}19)$$

$$\frac{u_x}{u_*} = \left(\frac{u_w}{u_*} - 5.75 \log m\right) + 5.75 \log \frac{r}{e} \quad [i, cc] \; (4\text{-}20)$$

By substituting A_r, which is not a constant, for the parenthetical expression, a simpler appearing equation is obtained:

$$\frac{u_x}{u_*} = A_r + 5.75 \log \frac{r_d}{e} \qquad [i, cc] \; (4\text{-}21)$$

The form of (4-31) is more easily handled than (4-20) in the treatment of experimental data. It is observed that (4-21) is of the same form as found for smooth pipes except that the first term of the right hand side is no longer constant.

Replacing r_d by y_d in Equation (4-21) would yield an expression suitable for analyzing two-dimensional flow between parallel plates.

4-5 Friction Factor

The maximum velocity in flow in a straight circular pipe is obtained from (4-21) as

$$\frac{u_{max}}{u_*} = A_r + 5.75 \log \frac{r_o}{e} \qquad [i, cc] \quad (4\text{-}22)$$

Noting that

$$\frac{u_{max} - U}{u_*} = D = \text{constant} \quad [i,u] \ (4\text{-}23)$$

also holds for flow in rough pipes with almost the same value for D as in the smooth case there is obtained

$$\frac{U}{u_*} = A_r - D + 5.75 \log \frac{r_o}{e} \quad [i,u] \ (4\text{-}24)$$

after substituting (4-23) in (4-22).

Since the resistance factor is given by

$$\sqrt{\frac{8}{\lambda}} = \frac{U}{u_*} \quad (4\text{-}25)$$

Equation (4-24) becomes

$$\frac{1}{\sqrt{\lambda}} = \frac{A_r - D}{\sqrt{8}} + \frac{5.75}{\sqrt{8}} \log \frac{r_o}{e} \quad [i,u] \ (4\text{-}26)$$

In flow between parallel plates r_o would be replaced by y_o.

4-6 Experimental Results for Flow in Circular Pipes

Again Nikuradse has provided the experimental data for proof of the theory. In his analysis, values of A_r were computed from Equation (4-26) using resistance data and a value of D as 3.75. The results of his calculations are plotted in Figure 4-3 against the logarithm of the dimensionless parameter $u_* e/\nu$ which is not directly dependent upon Re. The value of A_r is seen to increase to a maximum and then decrease slightly to maintain a constant value. The range of the constant A_r value is the region of wholly "rough" flow where the resistance is independent of Re and proportional to the square of the velocity. The value of A_r in the latter zone is taken as 8.48. Substituting A_r and D in (4-21) and (4-26) gives the equations:

$$\frac{u_x}{u_*} = 8.48 + 5.75 \log \frac{r_d}{e} \qquad [i, cc] \ (4\text{-}27)$$

$$\frac{1}{\sqrt{\lambda}} = 1.67 + 2.03 \log \frac{r_o}{e} \qquad [i, cc] \ (4\text{-}28)$$

Actually Nikuradse presented the resistance equation as

$$\frac{1}{\sqrt{\lambda}} = 1.74 + 2 \log \frac{r_o}{e} \qquad [i, cc] \ (4\text{-}29)$$

The straight line in Figure 4-3 shows the velocity distribution in wholly "rough" flow as obtained from Equation (4-27). Similarly the line in Figure 4-5 is obtained from (4-28) the resistance equation.

The first term of the right hand side of Equation (4-26) is plotted in Figure 4-6 as a horizontal straight line. Both Figures 4-3 and 4-6 show that for geometrically similar roughness, similar flow conditions are obtained over a wide range of relative roughness.

In Figures 4-3 and 4-6 diagonal straight lines are included showing the relationships of the various parameters in "smooth" flow. The lines have been developed by transforming the equations of "smooth" flow. The transform for Figure 4-3 is obtained from

$$\frac{u_x}{u_*} = 5.5 + 5.75 \log \frac{r_d u_*}{V} \quad [i,cc] \; (4-29)$$

Here the substitution

$$\log \frac{r_d u_*}{V} = \log \frac{r_d}{e} \cdot \frac{e u_*}{V} = \log \frac{r_d}{e} + \log \frac{e u_*}{V} \quad (4-30) \; [i,cc]$$

is made to give

$$A_r = \frac{u_x}{u_*} - 5.75 \log \frac{r_d}{e} = 5.5 + 5.75 \log \frac{e u_*}{V} \; [i,cc] \; (4-31)$$

which is the equation of the diagonal straight line. In the same manner the diagonal straight line in Figure 4-6 is obtained. The resistance equation for "smooth" flow is, according to Bakhmeteff

$$\frac{1}{\sqrt{\lambda}} = 2 \log \frac{r_o u_*}{V} + 0.5 \qquad [i,cc] \; (4-32)$$

By the manipulation noted in (4-30), (4-32) becomes

$$\frac{1}{\sqrt{\lambda}} - 2 \log \frac{r_o}{e} = 2 \log \frac{e u_*}{V} + 0.5 \quad [i,cc] \; (4-33)$$

which is the straight line.

Actual velocity distributions are shown in Figure 4-7 for different relative pipe roughnesses. The data for flow in smooth pipes are included. In Figure 4-8 the data are replotted using u_x/u_* as the dependent variable in in place of $\frac{u_x}{u_{max}}$ used in Figure 4-6 .

The empirical coefficients that have been obtained from Nikuradse's work with rough pipes are of course limited in application to flow equations where the geometrical similarity of roughness is maintained. Other types of roughness no doubt will yield different sets of coefficients.

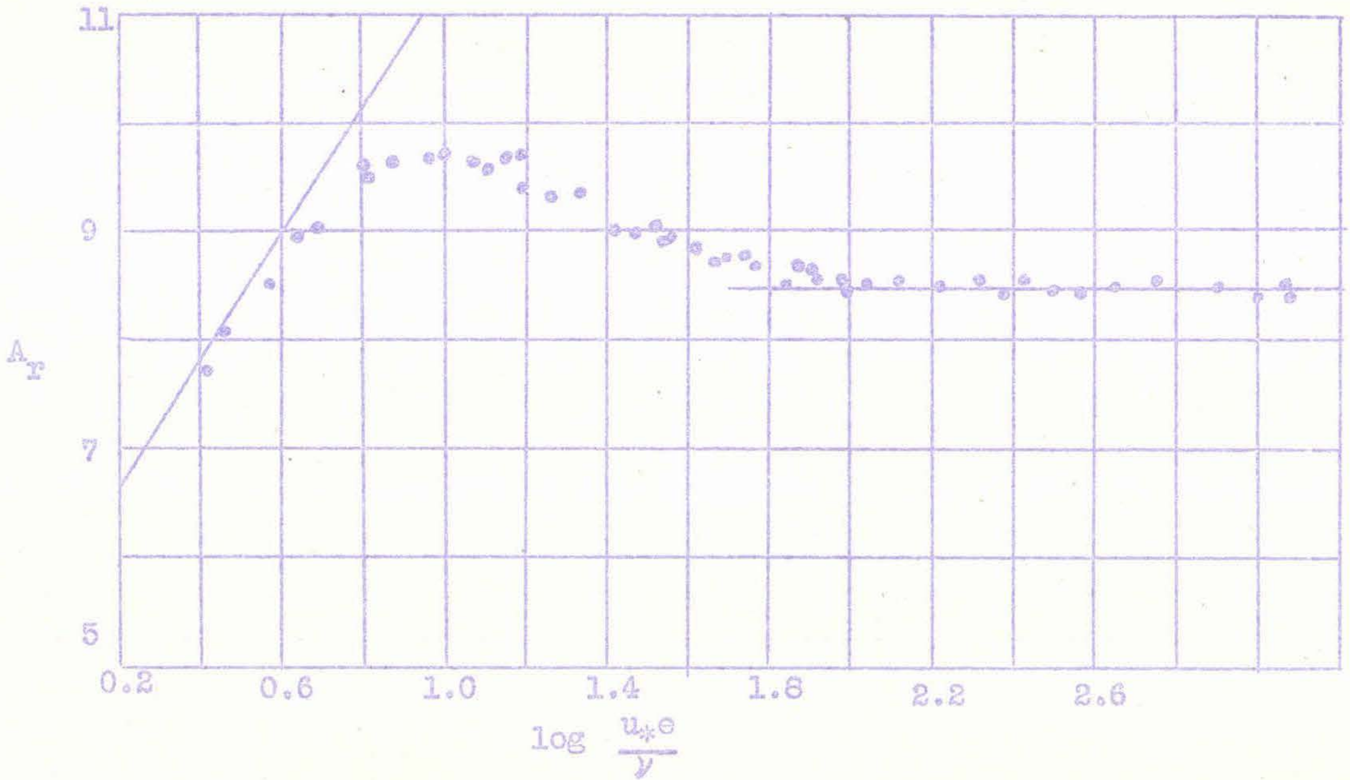

Figure 3.

(After Nikuradze, in Forschungsheft V.D.I. 361.)

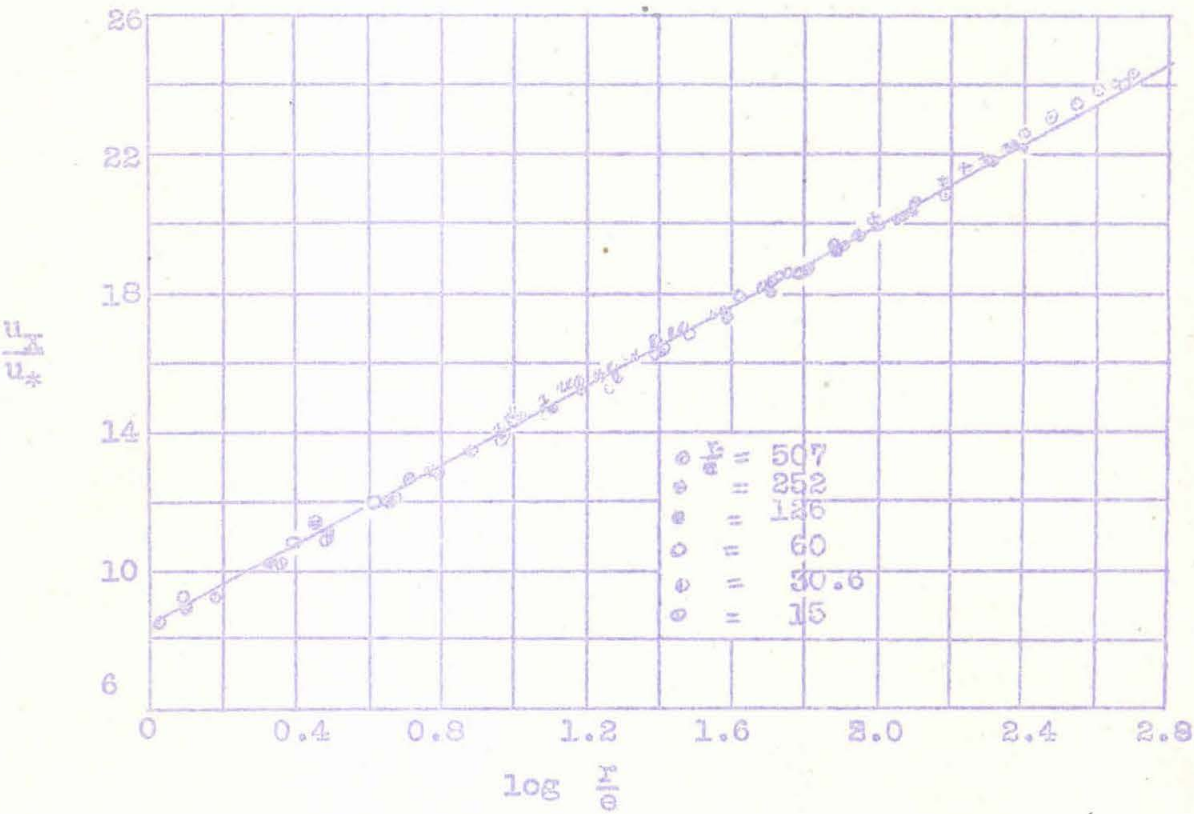

Figure 4.

(After Nikuradze, in Forschungsheft V.D.I. 361.)

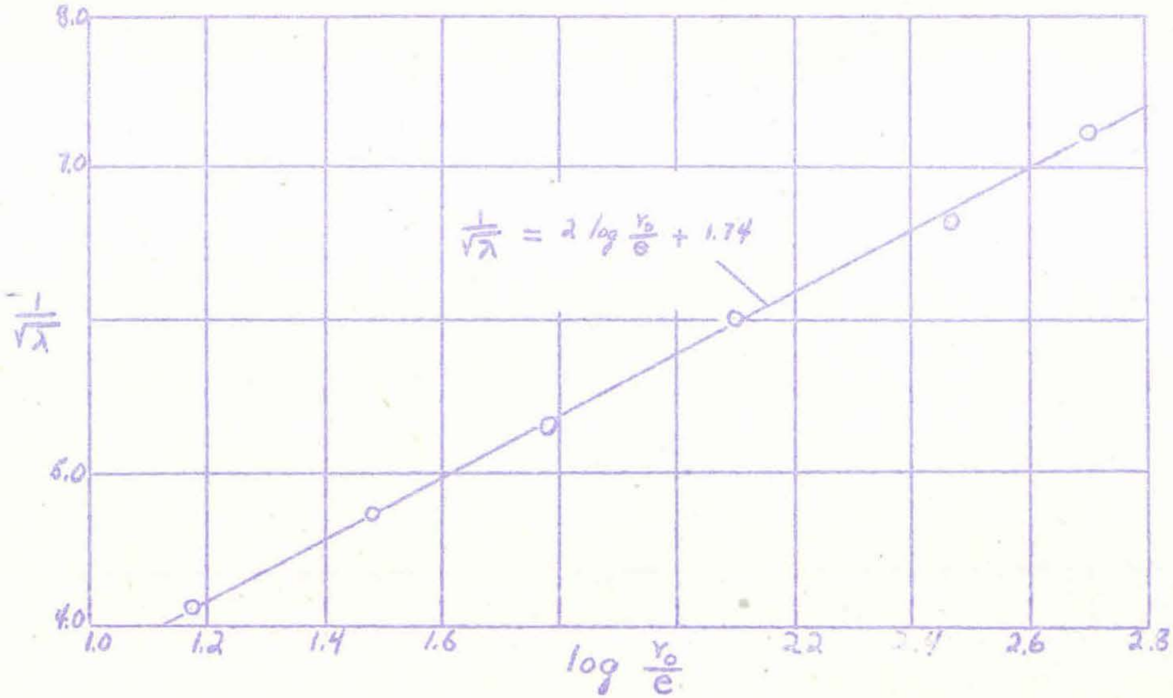

Figure 5

(After Nikuradze, in Forschungsheft V.D.I. 361.)

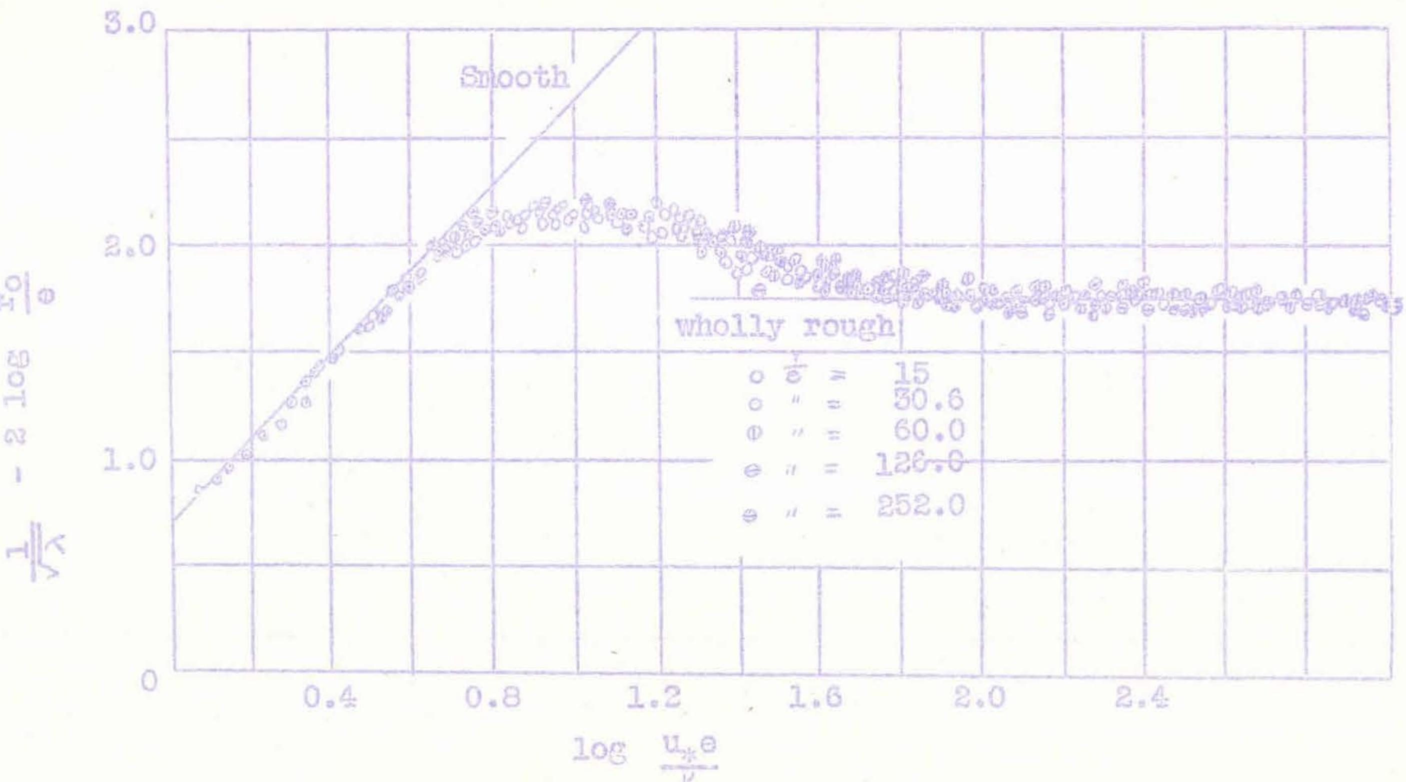

Figure 6. Prandtl's generalized roughness function. (After Nikuradze, in Forschungsheft V.D.I. 361.)

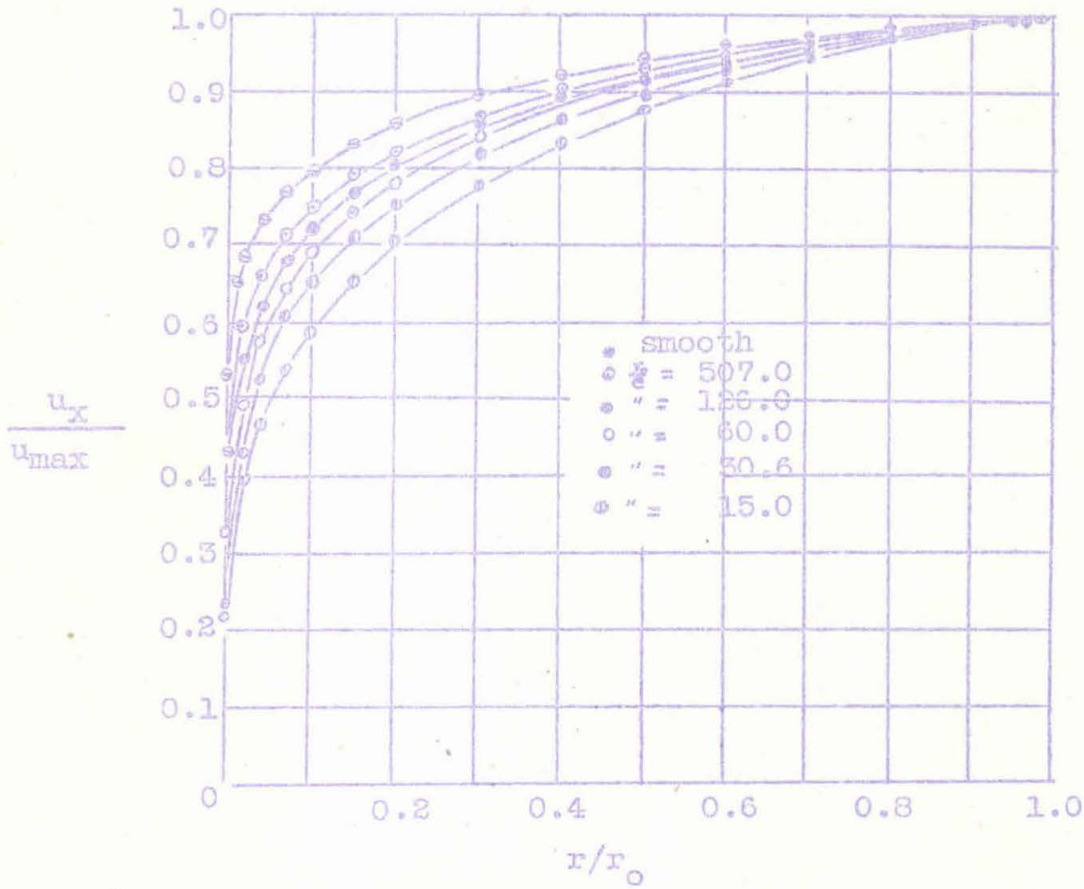

Figure 7. Velocity distribution for smooth and rough pipes. (After Nikuradze, in Forschungsheft V.D.I. 361.)

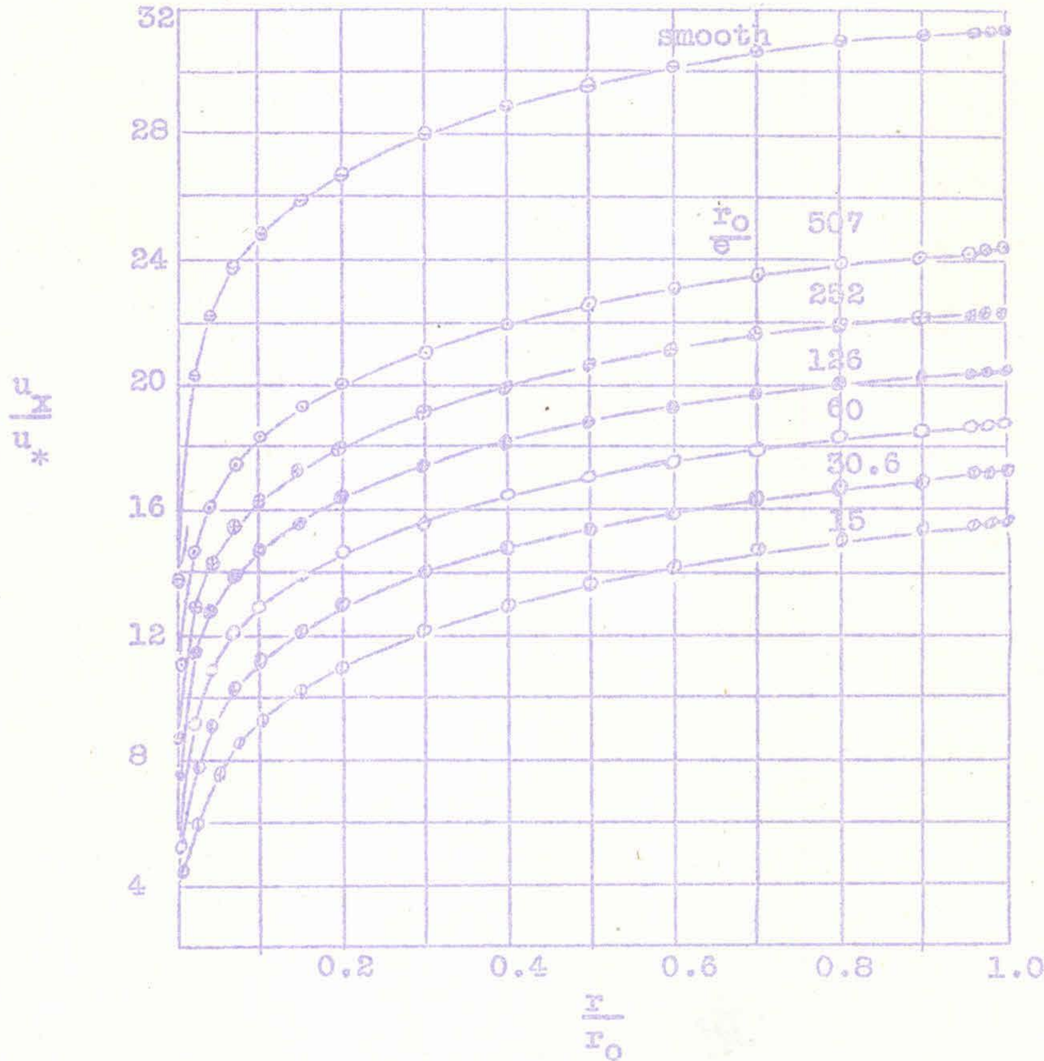

Figure 8. Velocity distribution in smooth and rough pipes. (After Nikuradze, in Forschungsheft V.D.I. 361.)

CHAPTER V

GENERAL EQUATIONS OF FLUID MOTION

INTRODUCTION

The general equations of fluid motion will be derived in this chapter. One equation expresses the conservation of matter and is known as the Equation of Continuity, and there are three equations which express the conservation of momentum in the three coordinate directions and are known the Stokes-Navier Equations.

Because of the large number of relations to be derived, it is not practicable to give many applications of these equations. However, many references are given to applications, and in Chapters IX & X, where the equation expressing the conservation of energy is derived , which derivation completes the analysis of the behavior of a homogeneous fluid, further applications will be given.

The reader is, therefore, requested to be patient.

Note: In this chapter, the previous convention that an unmodified velocity, etc. signifies an average velocity (See Chapter I) will be altered to the convention that an unmodified velocity signifies an instantaneous velocity. Average velocities, etc. will be designated by a bar over the symbol.

CHAPTER V

GENERAL EQUATIONS OF FLUID MOTION

The first relationship to be established in this chapter is that expressing the conservation of matter.

5-1 Equation of Continuity

Because of the law of the conservation of matter, the time rate of change of mass within an element of volume of a homogeneous flowing fluid is directly related to the rate at which mass flows into the volume element across the bounding surfaces of the element.

Consider a volume element with edges parallel to the X-Y-Z Cartesian coordinate axes, of lengths dx, dy, dz, and with center at (x,y,z). (Fig 5-1)

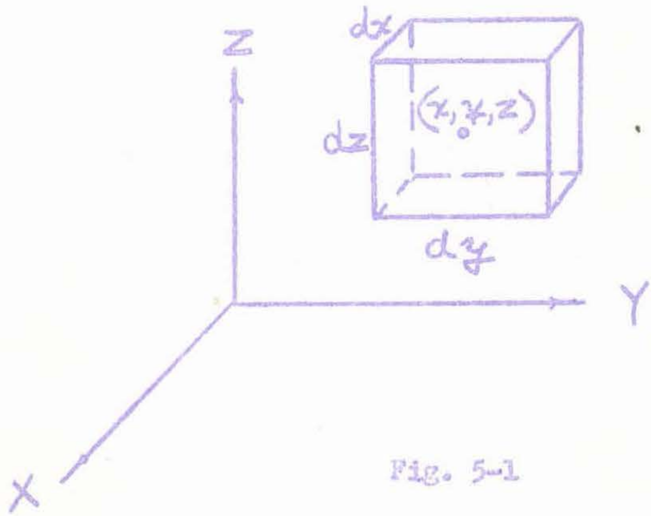

Fig. 5-1

The Cartesian axes will be considered as fixed relative to the observer as will the volume element itself. If the components of the velocity of the fluid at (x, y, z) are u_x, u_y, u_z in the X,Y,Z directions respectively, then the mass of the fluid flowing out through the right dxdz face in unit time is

$$\left(\rho u_y + \left(\frac{\partial(\rho u_y)}{\partial y} \cdot \frac{dy}{2}\right) dx\,dz\right.$$

and that flowing in through the left dxdz face in unit time is

$$\left(\rho u_y - \left(\frac{\partial(\rho u_y)}{\partial y} \cdot \frac{dy}{2}\right) dx\,dz\right.$$

The net gain of mass by flow through these two faces per unit time is therefore

$$-\left(\frac{\partial(\rho u_y)}{\partial y}\right)_{x,z,\theta} dx\,dy\,dz$$

Similarly the net gain in the X-direction is

$$-\left(\frac{\partial(\rho u_x)}{\partial x}\right)_{y,z,\theta} dx\,dy\,dz$$

and in the Z-direction

$$-\left(\frac{\partial(\rho u_z)}{\partial z}\right)_{x,y,\theta} dx\,dy\,dz$$

This rate of gain of mass by flow through the bounding surfaces of the volume element must equal the rate of increase of mass within the volume element, by the law of the conservation of matter (provided that no nuclear reactions or relativistic processes are occurring[1]).

The latter gain, however, is

$$\left(\frac{\partial \rho}{\partial \theta}\right)_{x,y,z} dx\,dy\,dz$$

Hence

$$\left(\frac{\partial \rho}{\partial \theta}\right)_{x,y,z} + \left(\frac{\partial(\rho u_x)}{\partial x}\right)_{y,z,\theta} + \left(\frac{\partial(\rho u_y)}{\partial y}\right)_{x,z,\theta} + \left(\frac{\partial(\rho u_z)}{\partial z}\right)_{x,y,\theta} = 0 \qquad [-l, m] \quad (5-1)$$

This equation is not restricted to systems of constant composition, if the total mass and not the mass of a component is considered. Supplementary relations are necessary in case diffusion processes are occurring in order to describe the behavior of each component of the fluid. (See Chapter XIV.) Chemical reactions introduce an additional complication and are not considered in this discussion. (See The Thermodynamics of Irreversible Processes, II Fluid Mixtures, Carl Eckart, Phys. Rev. 58, 269-275, (1940).)

It is sometimes convenient in theoretical treatments of fluid motion to introduce the concept of a pipe of infinitesimal cross section which has one end within a fluid element of volume. If the pipe adds material to the element it is known as a source; if it removes material,

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1 See C. Eckart, The Thermodynamics of Irreversible Processes, III. The Relativistic Theory of the Simple Fluid, Phys. Review 58, 919-924 (1940).

a sink; and at the point where the pipe ends there is an infinite discontinuity in the density if the material is added or withdrawn at a finite rate. In such a case, Equation (5-1) must be modified by the addition to the right side of a term expressing the instantaneous rate of addition of material at the point (x, y, z).

Equation (5-1) may be written in another useful form by expanding the partial derivatives of the products of the density and the velocity components:

$$\frac{\partial \rho}{\partial \theta} + u_x \frac{\partial \rho}{\partial x} + u_y \frac{\partial \rho}{\partial y} + u_z \frac{\partial \rho}{\partial z} + \rho \left\{ \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right\} = 0 \qquad [l, m] \quad (5\text{-}2)$$

Or

$$\frac{d\rho}{d\theta} = -\rho \left[\left(\frac{\partial u_x}{\partial x}\right)_{y,z,\theta} + \left(\frac{\partial u_y}{\partial y}\right)_{x,z,\theta} + \left(\frac{\partial u_z}{\partial z}\right)_{x,y,\theta} \right] \qquad [l, m] \quad (5\text{-}3)$$

Since

$$\frac{d\rho}{d\theta} = \frac{\partial \rho}{\partial \theta} + \frac{\partial \rho}{\partial x} \cdot \frac{dx}{d\theta} + \frac{\partial \rho}{\partial y} \cdot \frac{dy}{d\theta} + \frac{\partial \rho}{\partial z} \cdot \frac{dz}{d\theta}$$

$$= \frac{\partial \rho}{\partial \theta} + u_x \frac{\partial \rho}{\partial x} + u_y \frac{\partial \rho}{\partial y} + u_z \frac{\partial \rho}{\partial z} \qquad (5\text{-}4)$$

if the density is considered to be a function of time and position only, by the ordinary rule for the expansion of a total derivative in terms of its independent variables.

<u>Note:</u> Many authors in the field of fluid mechanics use the symbol $\frac{D}{D\Theta}$ for $\frac{d}{d\Theta}$ in Equations (5-3) and (5-4) because they desire to distinguish this total derivative from others which involve other independent variables, e.g. P. and T in a thermodynamic equation. In this connection, another interpretation may be placed on Equation (5-4). If the center of an element of volume of a fluid moves from (x,y,z) to (x+dx, y+dy, z+dz) in the time $d\Theta$ Equation (5-4) expresses the rate of change of density with time as the fluid flows along.

Equation (5-1) or (5-3) is known as the Equation of Continuity, and it may also be derived by considering the distortion of a fluid element of volume which moves with the flow[1].

Further, it may be derived without using approximations and without requiring any special orientation and shape for the fluid element of volume. This latter derivation requires the use of an important theorem of calculus known variously as Gauss' or Green's theorem. (In this discussion it will only be called Gauss' theorem to avoid confusion.) Since this theorem will be needed frequently later, it will be given here and illustrated by the derivation.

The theorem states that if ξ_x, ξ_y, and ξ_z are the components of a quantity in the X, Y, and Z-directions respectively, if ξ_x, ξ_y, and ξ_z are finite and continuous functions of x, y, and z throughout some region of space R as are their first derivatives with respect to x, y, and z respectively, then:

$$\iiint \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \right) dx\,dy\,dz = \iint (X\,dy\,dz + Y\,dx\,dz + Z\,dx\,dy) \qquad 5\text{-}5$$

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1 Lamb, H., Hydrodynamics 6th Edition, Dover (1945) p-4. Hereafter called simply Lamb.

where the triple integral is taken throughout the region R, and the double integral is taken over the entire surface of the region R. The former integral is known as a volume integral, and the latter a surface integral. The theorem is proved in most advanced calculus texts.[1]

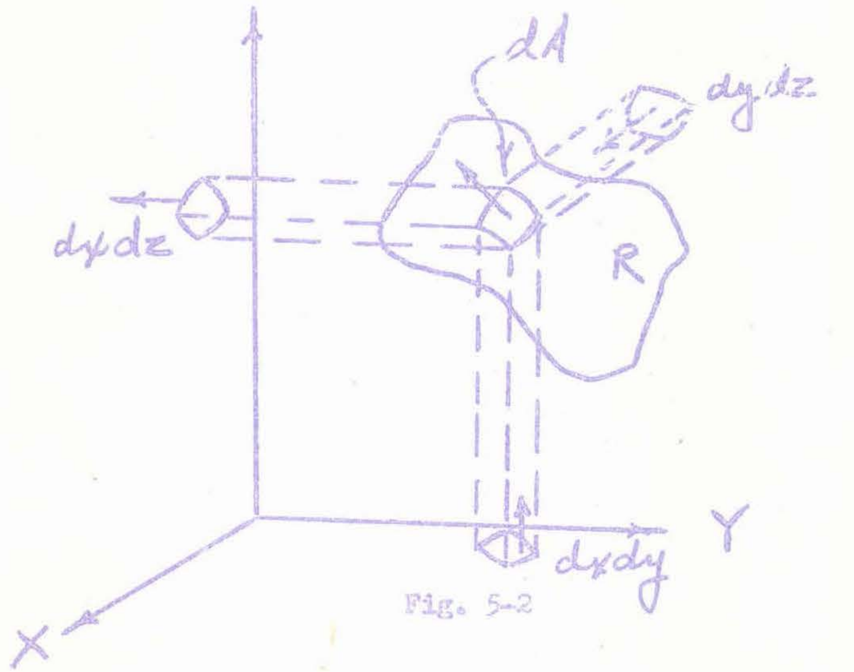

Fig. 5-2

There are several conventions which must be observed in connection with the double or surface integral. dxdy, dydz and dxdz are the projections of the element of area dA of the surface of the region R on the XY, YZ, and XZ planes respectively. The element of area dA is considered to have a positive and a negative direction associated with it; the positive direction is taken as that direction perpendicular to the element of surface dA of the region R and going away from it, i.e. is the outward normal to the element of surface dA. The projected elements of area dxdy, dydz, and dxdz also have

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1 Wilson, E. B., Advanced Calculus Ginn and Company (1912), p-341. Hereafter called Wilson.
Burrington and Torrance Higher Mathematics, Mc Graw Hill Company (1939), p-264. Hereafter called Burrington.

positive directions which are the same as those of the outward normal projected in the Z, X, Y directions respectively. Thus in Figure 5-2, the positive direction of dydz is the +X-direction; dxdy, the +Z; and dxdz, the - Y-direction. If the direction in which ξ_x is assumed positive is the same as the positive direction of dydz then ξ_x dydz is taken as positive in Equation (5-5), otherwise negative; and similarily for the other components and projected surface elements.

To derive the Equation of Continuity, let ξ_x be the mass rate of flow per unit area in the X-direction, i.e. $\xi_x = \rho u_x$ and similarily $\xi_y = \rho u_y$, and $\xi_z = \rho u_z$. Then from Gauss' theorem Equation (5-5),

$$\iiint\left[\frac{\partial(\rho u_x)}{\partial x}+\frac{\partial(\rho u_y)}{\partial y}+\frac{\partial(\rho u_z)}{\partial z}\right]dx\,dy\,dz = \iint\left[\rho u_x\,dydz+\rho u_y\,dxdz+\rho u_z\,dxdy\right] \quad [h] \quad (5\text{-}6)$$

Now the surface or double integral gives the rate at which mass flows out through the surface of the region R in the flowing fluid which is equal to the rate of decrease of mass within the surface. Hence

$$-\iiint\frac{\partial\rho}{\partial\theta}\,dx\,dy\,dz = \iint\left[\rho u_x\,dydz+\rho u_y\,dxdz+\rho u_z\,dxdy\right] \quad [h, \underline{m}] \quad (5\text{-}7)$$

Or, combining Equations (5-6) and (5-7),

$$\iiint\left[\frac{\partial\rho}{\partial\theta}+\frac{\partial\rho u_x}{\partial x}+\frac{\partial\rho u_y}{\partial y}+\frac{\partial\rho u_z}{\partial z}\right]dx\,dy\,dz = 0 \quad [h, \underline{m}] \quad (5\text{-}8)$$

The restriction of homogeneity is equivalent to the requirement that ξ_x and $\frac{\xi_x}{\partial x}$, etc. are finite and continuous.

Since the size and shape of the region R can be varied arbitrarily, the integrand must vanish for each point in the flow in order to satisfy Equation (5-8). Hence Equation (5-1) is obtained.

In a volume element of a homogeneous fluid, the pressure and temperature have been assumed to be definable[1] and continuous; that is, they may be expressed as linear functions of the dimensions of the volume element. The restrictions on the use of the concepts of pressure and temperature will be discussed in Chapter IX. Thus for the fluid there will be, in general, an equation of state of the form

$$\rho = \rho(P, T) \qquad [0] \ (5\text{-}9)$$

Here the possibility of diffusion and variation of composition is excluded. Equation (5-3) may be written

$$\left(\frac{\partial \rho}{\partial T}\right)_P \frac{dT}{d\theta} + \left(\frac{\partial \rho}{\partial P}\right)_T \frac{dP}{d\theta} + \rho\left[\left(\frac{\partial u_x}{\partial x}\right)_{y,z,\theta} + \left(\frac{\partial u_y}{\partial y}\right)_{x,z,\theta} + \left(\frac{\partial u_z}{\partial z}\right)_{x,y,\theta}\right] = 0 \qquad [m, l] \ 5\text{-}10$$

to emphasize that ρ can be considered a thermodynamic function of P and T alone. $\frac{d}{d\theta}$ has the same significance as in Equation (5-3).

For a liquid which is both incompressible and thermally inexpansible, i.e. for which

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1. See Chapter I, Section 4.

$$\left(\frac{\partial \rho}{\partial P}\right)_T = \left(\frac{\partial \rho}{\partial T}\right)_P = 0 \text{ or } \rho = \rho_0 \qquad [0] \; (5\text{-}11)$$

from Equation (5-3) or (5-10),

$$\left(\frac{\partial u_x}{\partial x}\right)_{y,z,\theta} + \left(\frac{\partial u_y}{\partial y}\right)_{x,z,\theta} + \left(\frac{\partial u_z}{\partial z}\right)_{x,y,\theta} = 0 \qquad [0, \underline{\underline{m}}, h] \; (5\text{-}12)$$

The left hand side of Equation (5-12) is known as the divergence of the velocity so that Equation (5-3) is sometimes written in shorthand fashion as

$$\frac{d\rho}{d\theta} + \rho \, \text{div} \, u = 0 \qquad [h, \underline{\underline{m}}] \; (5\text{-}13)$$

and Equation (5-12) becomes

$$\text{div} \, u = 0 \qquad [0, h, \underline{\underline{m}}] \; (5\text{-}14)$$

By considering a volume element of constant mass it can be seen from Equation (5-13) by dividing by ρ and substituting $\underline{\underline{m}}_0/V$ for ρ that div u represents the relative rate of expansion of the fluid.

The Equation of Continuity has been derived in the Cartesian coordinate system. In applications this system is useful when any or all of the surfaces bounding the flow are planes. However, cylindrical, spherical, etc. coordinate systems are more useful in case the bounding surfaces are cylinders, spheres, etc. The Equation of Continuity may be rederived for each coordinate system, the Equation in Cartesian coordinates may be transformed to the desired system by means of the equations relating the coordinates[1], or the equation may be derived in a general form Equation (5-13)),[2] and the special results for each coordinate system tabulated. The techniques for obtaining each particular result are explained in most vector analysis texts.[3] Extensive tabulations of the results are given in Margenau and Murphy, and Adams.[4]

The results will be given here for the important cases of cylindrical and spherical coordinates

Cylindrical Coordinates

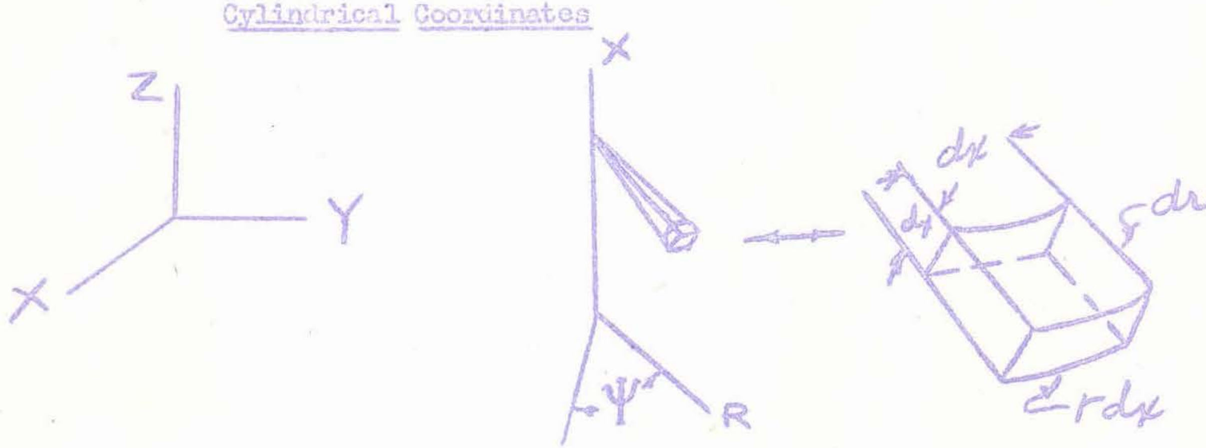

Fig. 5-3

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1 Wilson p-98
2 L. M. Milne-Thomson, Theoretical Hydrodynamics, Macmillan and Company (1938) p-64. Hereafter called Milne-Thomson.
3 Phillips, H. B., Vector Analysis, John Wiley and Sons (1933). Hereafter Phillips; Gibbs-Wilson, Vector Analysis Yale University Press (1925).
4 Margenau, H. and Murphy, G. M., The Mathematics of Physics and Chemistry, D. Van Nostrand Company (1943) Chapter 5. Hereafter Margenau and Murphy. Smithsonian Mathematical Formulae, Editor E. P. Adams, Washington (1922). Hereafter Adams.

$$V\left(\frac{\partial \rho}{\partial \theta}\right)_{r,\psi,x}+\frac{1}{r}\left[\left(\frac{\partial(r\rho u_r)}{\partial r}\right)_{\psi,x,\theta}+\left(\frac{\partial(\rho u_x)}{\partial x}\right)_{r,x,\theta}+\left(\frac{\partial(r\rho u_x)}{\partial x}\right)_{r,\psi,\theta}\right]=0$$

[m, h] (5-15)

is the Equation of Continuity in cylindrical coordinates, where R is the radial direction, Ψ the azimuthal, and X the axial direction (Note: in many books on mathematical physics Z is taken as the axial direction and Θ as the azimuthal). u_r, u_ψ, and u_x are the components of the velocity in the radial, azimuthal and axial directions respectively.

This form of Equation (5-1) is especially useful when considering flow in cylinders (e.g. pipes) ie. where one of the flow boundaries can be taken as $r = r_0$, a constant.

Spherical Coordinates

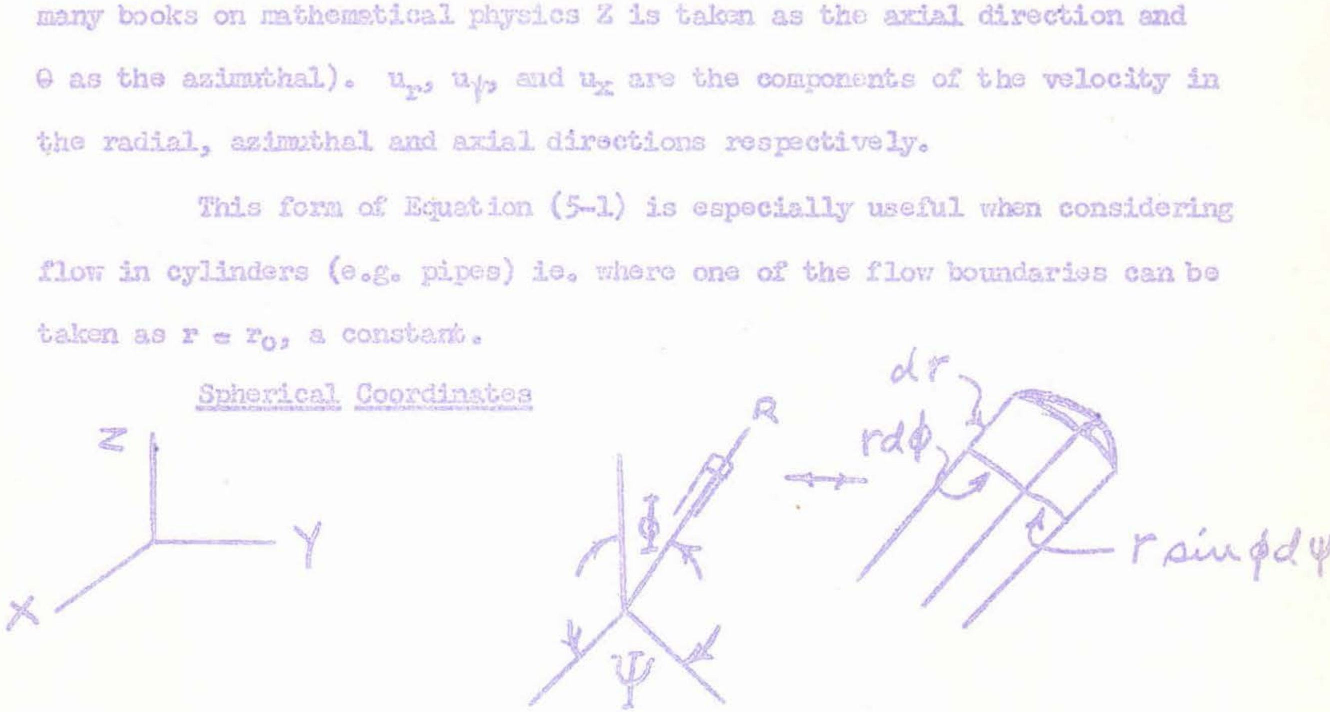

Fig. 5-4

The Equation of Continuity in spherical coordinates is

$$\left(\frac{\partial\rho}{\partial\theta}\right)_{r,\psi,\phi}+\frac{1}{r^2\sin\phi}\left\{\left(\frac{\partial(\rho u_r r^2\sin\phi)}{\partial r}\right)_{\psi,\phi,\theta}+\left(\frac{\partial(\rho u_\phi r\sin\phi)}{\partial\phi}\right)_{r,\psi,\theta}+\left(\frac{\partial(\rho u_\psi r)}{\partial\psi}\right)_{r,\phi,\theta}\right\}=0$$

[m, h] 5-15a

where R is the radial direction, Ψ the azimuthal, and Φ the colatitude direction, and u_r, u_ψ, u_ϕ are the components of the velocity in the radial, azimuthal and colatitude directions. This form of Equation (5-1) is most useful when one of the surfaces bounding the flow is a sphere or a portion of one ie. for which $r = r_0$, a constant.

5-2 Boundary Conditions for the Equation of Continuity

The only boundaries to a homogeneous flowing fluid which will be considered here are solid walls or imaginary surfaces in the fluid itself. The imaginary surfaces may have any properties it is desired to ascribe to them, but the situation at a solid-fluid interface may be more complicated than it is possible to consider in this discussion. Therefore it is assumed that there are no solution (or crystallization) processes occurring at the interface between the solid walls and the fluid and that the walls are continuous, smooth, and non-porous. Thus it may be assumed farther that the interfacial energy is negligible compared to the internal energy of the fluid.

As mentioned in Chapter I, the relative velocity between a real fluid and a solid wall is generally negligible (see Chapter IX for a more precise treatment); consequently, if u' is the velocity of a point of the surface of the solid (for example a point on the surface of a stirrer blade) and u the velocity of the fluid in contact with that point,

$$u' - u = 0 \qquad [o] \qquad (5\text{-}16)$$

Thus if $u' = 0$, u must be zero.

An important branch of theoretical fluid dynamics considers the behavior of a "perfect" fluid i.e. a fluid which has no viscosity. In place of Equation (5-16) equations may be derived expressing the fact that the relative velocity of the fluid normal to the wall is zero[1], but, in general, for this hypothetical case the relative tangential velocity need not be.

The Equation of Continuity is not of much value by itself since it introduces 5 unknown quantities, even for a system of constant composition (say P, T, u_x, u_y, u_z), which are functions of the 4 independent variables x, y, z, and θ, say. No illustration of its application will be given since it provides only one relation among these five unknown quantities; and hence to obtain a definite solution, four of the unknowns must be eliminated as variables by rather artificial and forced assumptions.

Consequently the discussion of methods of solution of partial differential equations such as Equation (5-1) together with their boundary conditions such as Equation (5-16) will be deferred until more relations among the unknown variables have been derived.

Three more equations relating these variables can be found by the law of the conservation of momentum; or its equivalent, a force balnce, which can be calculated by adding all the forces acting on a region or volume element of a fluid.

Because of the complexity of the situation, it is desirable to discuss separately each type of force which acts on the fluid. In general the acceleration of the fluid is produced by external forces such as gravitational and electric fields and by forces exerted by adjacent portions

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1 See Lamb and Milne-Thomson.

of the fluid on one another.

5-3 Acceleration of the Flowing Fluid

The acceleration of a flowing fluid at any particular point in the flow (eg. the center of an element of volume) is given by the rate of change of the velocity at that point by Newton's second law of motion, if the portion of the fluid under consideration is taken in such a manner that its mass is constant.

Thus

$$a_x = \frac{du_x}{d\theta} = \left(\frac{\partial u_x}{\partial \theta}\right)_{x,y,z} + \left(\frac{\partial u_x}{\partial x}\right)_{y,z,\theta} \frac{dx}{d\theta} + \left(\frac{\partial u_x}{\partial y}\right)_{x,z,\theta} \frac{dy}{d\theta} + \left(\frac{\partial u_x}{\partial z}\right)_{x,y,\theta} \frac{dz}{d\theta} \qquad (5\text{-}17)$$

Or

$$a_x = \frac{du_x}{d\theta} = \left(\frac{\partial u_x}{\partial \theta}\right)_{x,y,z} + u_x\left(\frac{\partial u_x}{\partial x}\right)_{y,z,\theta} + u_y\left(\frac{\partial u_x}{\partial y}\right)_{x,z,\theta} + u_z\left(\frac{\partial u_x}{\partial z}\right)_{x,y,\theta} \qquad (5\text{-}18)$$

Similarily for the components of the acceleration of the fluid in the other coordinate directions

$$a_y = \frac{du_y}{d\theta} = \left(\frac{\partial u_y}{\partial \theta}\right)_{x,y,z} + u_x\left(\frac{\partial u_y}{\partial x}\right)_{y,z,\theta} + u_y\left(\frac{\partial u_y}{\partial y}\right)_{x,z,\theta} + u_z\left(\frac{\partial u_y}{\partial z}\right)_{x,y,\theta} \qquad (5\text{-}19)$$

and

$$a_z = \frac{du_z}{d\theta} = \left(\frac{\partial u_z}{\partial \theta}\right)_{x,y,z} + u_x\left(\frac{\partial u_z}{\partial x}\right)_{y,z,\theta} + u_y\left(\frac{\partial u_z}{\partial z}\right)_{x,z,\theta} + u_z\left(\frac{\partial u_z}{\partial z}\right)_{x,y,\theta} \qquad (5\text{-}20)$$

These equations may be written in several other forms, some of which will be given later (Sect.) which are more convenient for particular problems.

5-4 External Forces Acting on a Flowing Fluid

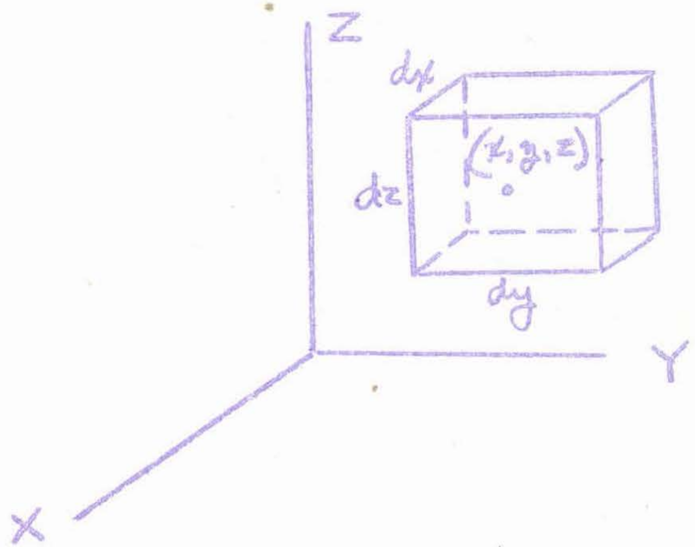

Fig. 5-5

Because of the possible influence of magnetic and electrical fields and the usual influence of gravitational fields on the matter of a flowing fluid, there may be external forces acting on an element of volume of the fluid such as that shown in Figure 5-5 of magnitude

$$\phi_x \rho dxdydz, \quad \phi_y \rho dxdydz, \text{ and } \phi_z \rho dxdydz$$

in the X,Y, and Z-directions respectively. ϕ_x is thus the acceleration in the X-direction produced by the external fields, etc.

If the fields are conservative, the forces are derivable from a potential,[1] ie. there exists a function Ω_x such that

$$\Phi_x = -\left(\frac{\partial \Omega}{\partial x}\right)_{y,z,\theta} \qquad [0] \quad 5\text{-}21$$

$$\Phi_y = -\left(\frac{\partial \Omega}{\partial y}\right)_{x,z,\theta} \qquad [0] \quad 5\text{-}22$$

$$\Phi_z = -\left(\frac{\partial \Omega}{\partial z}\right)_{x,y,\theta} \qquad [0] \quad 5\text{-}23$$

The minus sign is a convention adopted from the study of electricity. Thus for pure gravitational fields

$$\Omega = g\,h \qquad [0] \quad 5\text{-}24$$

where h is the vertical distance above a reference plane and g is the acceleration caused by gravity, so that Equations (5-21), (5-22), and (5-23) become

---

1 Page, L., Introduction to Theoretical Physics, D. Van Nostrand Company, (1935). Hereafter Page.

$$\Phi_x = -g\left(\frac{\partial h}{\partial x}\right)_{y,z} \qquad [0]\ 5\text{-}25$$

$$\Phi_y = -g\left(\frac{\partial h}{\partial y}\right)_{x,z} \qquad [0]\ 5\text{-}26$$

$$\Phi_z = -g\left(\frac{\partial h}{\partial z}\right)_{x,y} \qquad [0]\ 5\text{-}27$$

Electric and magnetic fields[1] are rarely of technical importance in the case of one component systems, though the behavior of a jet of molten metal or fused salt under the influence of either or both types of external field would be a possible example. Chemical reactions usually accompany the electrolysis of multicomponent systems and so that case falls outside the scope of this discussion, and magnetic fields are rarely of technical importance even for multicomponent systems. Hence, unless stated to the contrary, external fields will always be taken to be pure gravitational fields.

5-5 Forces Acting on the Surface of a Portion of a Flowing Fluid

1 Page, L., Introduction to Theoretical Physics, D. Van Nostrand Company, (1935). Hereafter Page.

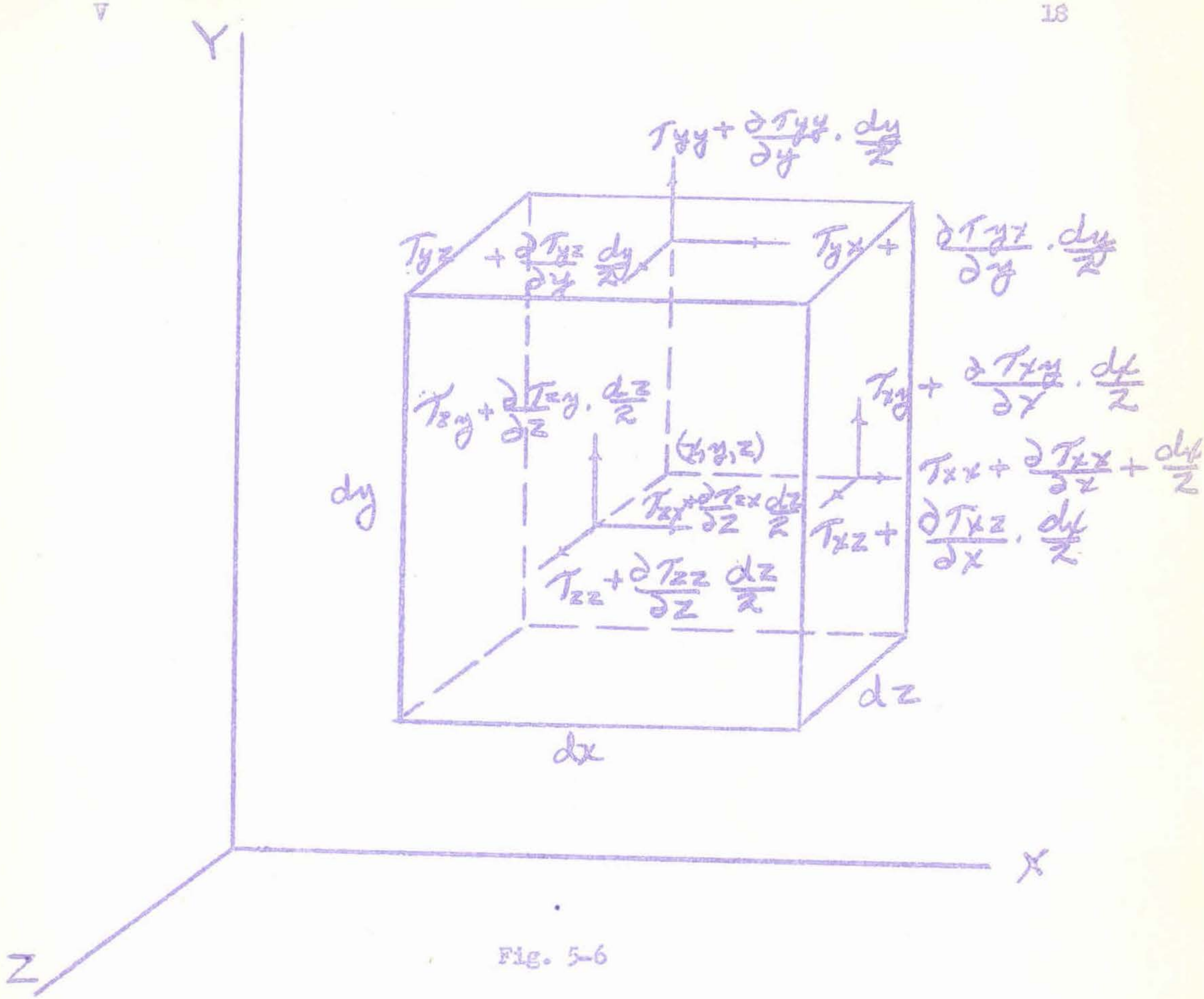

Fig. 5-6

The element of volume of the flowing fluid shown in Figure 5-6 has its center at (x, y, z) at time θ, and its edges of length dx, dy, and dz parallel to the respective coordinate axes. The surface stresses acting on three of the six faces are shown; the other three are similar, except that the direction of the stresses and the signs of their gradients are reversed. τ_{xx} represents the stress at the point (x, y, z) which acts on an element of surface passing through (x, y, z) perpendicular to the X-axis, and acts in the X-direction; τ_{xy}, the stress on an element of surface passing through (x, y, z) perpendicular to the X-axis, and acting in the Y-direction; etc.

The net force acting in the X-direction on the element of volume of Figure 5-6 is:

$$\left[\tau_{xx}+\left(\frac{\partial \tau_{xx}}{\partial x}\right)_{y,z,\theta}\frac{dx}{2}\right]dy\,dz+\left[-\tau_{xx}-\left(\frac{\partial(-\tau_{xx})}{\partial x}\right)_{yz\theta}\frac{dx}{2}\right]dy\,dz$$

$$+\left[\tau_{zx}+\left(\frac{\partial \tau_{zx}}{\partial z}\right)_{x,y,\theta}\frac{dz}{2}\right]dx\,dy+\left[-\tau_{zx}-\left(\frac{\partial(-\tau_{zx})}{\partial z}\right)_{xy\theta}\frac{dz}{2}\right]dx\,dy$$

$$+\left[\tau_{yx}+\left(\frac{\partial \tau_{yx}}{\partial y}\right)_{x,z,\theta}\frac{dy}{2}\right]dx\,dz+\left[-\tau_{yx}-\left(\frac{\partial(-\tau_{yx})}{\partial y}\right)_{x,z,\theta}\frac{dy}{2}\right]dx\,dz$$

$$=\left(\left(\frac{\partial \tau_{xx}}{\partial x}\right)_{y,z,\theta}+\left(\frac{\partial \tau_{yx}}{\partial y}\right)_{x,z,\theta}+\left(\frac{\partial \tau_{zx}}{\partial z}\right)_{x,y,\theta}\right)dx\,dy\,dz$$

Similarily the net force in the Y-direction is

$$\left(\left(\frac{\partial \tau_{xy}}{\partial x}\right)_{y,z,\theta}+\left(\frac{\partial \tau_{yy}}{\partial y}\right)_{x,z,\theta}+\left(\frac{\partial \tau_{zy}}{\partial z}\right)_{x,y,\theta}\right)dx\,dy$$

and in the Z-direction

$$\left(\left(\frac{\partial \tau_{xz}}{\partial x}\right)_{y,z,\theta}+\left(\frac{\partial \tau_{yz}}{\partial y}\right)_{x,z,\theta}+\left(\frac{\partial \tau_{zz}}{\partial z}\right)_{x,y,\theta}\right)dx\,dy\,dz$$

5-6 The Momentum Equations for a Flowing Fluid

Since by Newton's second law of motion[1], the time rate of change of momentum equals the net force acting on the element of volume of the flowing fluid, and since the element may be chosen so that its mass is constant, from Equation (5-18), (5-19), and (5-20) and the expressions for the external and surface forces derived in Sections 4 and 5 there is obtained

$$\frac{du_x}{d\theta}\rho\,dx\,dy\,dz = \Phi_x\rho\,dx\,dy\,dz + \left[\left(\frac{\partial T_{xx}}{\partial x}\right)_{y,z,\theta} + \left(\frac{\partial T_{yx}}{\partial y}\right)_{x,z,\theta} + \left(\frac{\partial T_{zx}}{\partial z}\right)_{x,y,\theta}\right]dx\,dy\,dz$$

for the X-direction and similarily for the other directions. Hence on dividing out dxdydz there is obtained

$$\frac{du_x}{d\theta} = \Phi_x + \frac{1}{\rho}\left[\left(\frac{\partial T_{xx}}{\partial x}\right)_{y,z,\theta} + \left(\frac{\partial T_{yx}}{\partial y}\right)_{x,z,\theta} + \left(\frac{\partial T_{zx}}{\partial z}\right)_{x,y,\theta}\right] \quad \text{5-28}$$

$$\frac{du_y}{d\theta} = \Phi_y + \frac{1}{\rho}\left[\left(\frac{\partial T_{xy}}{\partial x}\right)_{y,z,\theta} + \left(\frac{\partial T_{yy}}{\partial y}\right)_{x,z,\theta} + \left(\frac{\partial T_{zy}}{\partial z}\right)_{x,y,\theta}\right] \quad \text{5-29}$$

$$\frac{du_z}{d\theta} = \Phi_z + \frac{1}{\rho}\left[\left(\frac{\partial T_{xz}}{\partial x}\right)_{y,z,\theta} + \left(\frac{\partial T_{yz}}{\partial y}\right)_{x,z,\theta} + \left(\frac{\partial T_{zz}}{\partial z}\right)_{x,y,\theta}\right] \quad \text{5-30}$$

- -

1 For the sake of completeness it should be mentioned that again relativistic effects, radiation pressure, nuclear reactions, etc. are neglected. See Eckart's article referred to on 5-3.

These fundamental relations, which are known as the Momentum Equations, may also be derived for an arbitrary region of the flowing fluid by means of Gauss' theorem, which derivation will now be given.

Take an arbitrary region of the flowing fluid as in Figure 5-7 and apply Newton's second law of motion to it.

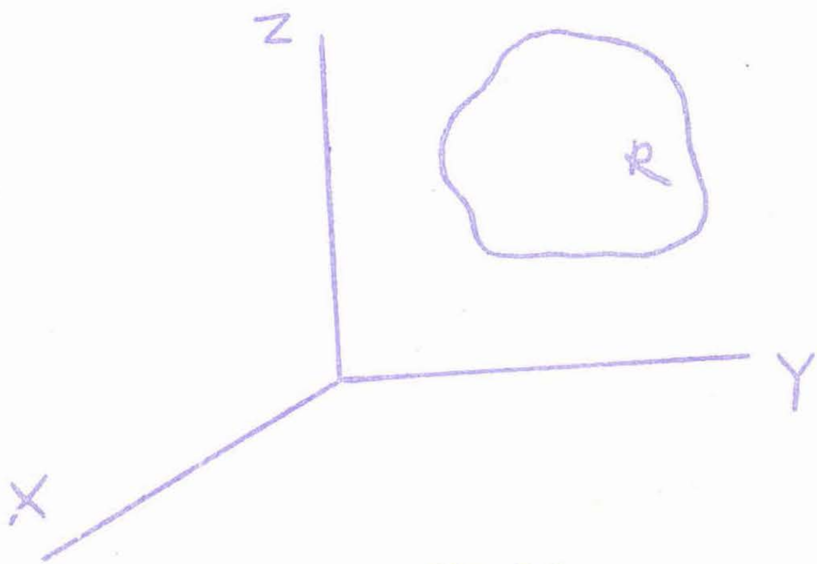

Fig. 5-7

The result for the X-direction is

$$\frac{d}{d\theta}\iiint \rho u_x \, dx\,dy\,dz = \iiint \Phi_x \rho \, dx\,dy\,dz + \iint \left[T_{xx}\,dy\,dz + T_{yx}\,dx\,dz + T_{zx}\,dx\,dy \right] \quad 5\text{-}31$$

where the volume integrals are extended over the entire region R and the surface integral over the entire surface of R, because ρu_x dxdydz is the momentum in the X-direction of an element of volume of the region R, $f_x\rho$dxdydz is the external force acting in the X-direction on an element of volume, and τ_{xx}dydz + τ_{yx}dxdz + τ_{zx}dydx is the surface force acting in the X-direction on an element of surface of the region R.

In order to evaluate the left-hand integral in Equation (5-31), it is best be evaluate first the integral

$$\frac{d}{d\theta}\iiint\limits_V \xi\, dx\,dy\,dz$$

where ξ is any continuous and finite function of x, y, z, and θ with continuous and finite first derivatives, and where the symbol V is written below the integral signs to emphasize that the integration is taken throughout the volume of the region R. By the definition of a total derivative

$$\frac{d}{d\theta}\iiint\limits_V \xi\, dx\,dy\,dz = \lim_{\Delta\theta\to 0}\frac{1}{\Delta\theta}\left\{\iiint\limits_{V(\theta+\Delta\theta)} \xi(\theta+\Delta\theta)\,dx\,dy\,dz - \iiint\limits_{V(\theta)} \xi(\theta)\,dx\,dy\,dz\right\} \qquad 5\text{-}32$$

since the volume of the region R as well as the value of ξ may vary with time. Add and subtract

$$\frac{1}{\Delta\theta}\iiint\limits_{V(\theta)} \xi(\theta+\Delta\theta)\,dx\,dy\,dz$$

to the right-hand side of Equation (5-32) inside the braces. Then

$$\frac{d}{d\theta}\iiint_V \xi\, dx\, dy\, dz = \lim_{\Delta\theta\to 0}\iiint_{V(\theta)}\left[\frac{\xi(\theta+\Delta\theta)-\xi(\theta)}{\Delta\theta}\right]dx\,dy\,dz$$

$$+\lim_{\Delta\theta\to 0}\frac{1}{\Delta\theta}\left[\iiint_{\Delta V(\theta)}\xi(\theta+\Delta\theta)\,dx\,dy\,dz\right] \qquad 5\text{-}33$$

where $\Delta V(\theta) = V(\theta + \Delta\theta) - V(\theta)$, the expansion of the region R in the time $\Delta\theta$. Since the integrating and limiting operations may be interchanged in the first term on the right side; by the definition of a partial derivative, this term is

$$\iiint_{V(\theta)}\left(\frac{\partial\xi}{\partial\theta}\right)_{xyz} dx\,dy\,dz$$

and the second term becomes, in the limit[1]

$$\xi \cdot \frac{dV}{d\theta}$$

that is, ξ times the rate of change of the volume of the region R. This term may, however, be rewritten[2] in terms of the rate of expansion of the surface of the region R for

$$\xi\frac{dV}{d\theta} = \iint_A \xi\left[u_x\,dy\,dz + u_y\,dx\,dz + u_z\,dx\,dy\right] \qquad 5\text{-}34$$

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1 See Wilson p-283.
2 P. Frank and R. von Mises Die Differential-und Integral gleichungen der Mechanik und Physik, 2nd Ed. Rosenberg (1943), Vol. 2, p-374. Hereafter Frank and von Mises.

since $u_x dydz$ gives the volume swept out by the X-component of the velocity and the element of area perpendicular to it in unit time, etc.

The symbol A indicates that the integration is taken over the entire surface of the region R. Thus Equation (5-32) becomes

$$\frac{d}{d\theta}\iiint_V \xi\, dx\,dy\,dz = \iiint_V \left(\frac{\partial \xi}{\partial \theta}\right)_{xyz} dx\,dy\,dz + \iint_A \xi (u_x\,dy\,dz + u_y\,dx\,dz + u_z\,dx\,dy) \qquad 5\text{-}35$$

The second term on the right-hand side may be transformed by means of Gauss' theorem, Equation (5-5), so that Equation (5-35) becomes

$$\frac{d}{d\theta}\iiint \xi\, dx\,dy\,dz = \iiint \left[\left(\frac{\partial \xi}{\partial \theta}\right)_{xyz} + \left(\frac{\partial(\xi u_x)}{\partial x}\right)_{y,z,\theta} + \left(\frac{\partial(\xi u_y)}{\partial y}\right)_{x,z,\theta} + \left(\frac{\partial(\xi u_z)}{\partial z}\right)_{x,y,\theta}\right] dx\,dy\,dz \qquad 5\text{-}36$$

Or, expanding the derivatives of the ξ-velocity products and remembering the expansion of a total derivative with respect to θ in x, y, z, and θ (Equation (5-4)),

$$\frac{d}{d\theta}\iiint_V \xi\, dx\,dy\,dz = \iiint_V \left[\frac{d\xi}{d\theta} + \xi\left[\left(\frac{\partial u_x}{\partial x}\right)_{y,z,\theta} + \left(\frac{\partial u_y}{\partial y}\right)_{x,z,\theta} + \left(\frac{\partial u_x}{\partial z}\right)_{x,y,\theta}\right]\right] dx\,dy\,dx \qquad 5\text{-}37$$

This result is that mentioned in the first paragraph after Equation (5-31). Therefore, returning to the evaluation of the left-hand integral in Equation (5-31), substitute $\rho u_x = \xi$ in Equation (5-37) and expand the derivative of the product, giving

$$\frac{d}{d\theta}\iiint \rho u_x \, dx\,dy\,dz = \iiint \left[u_x\left(\frac{\partial\rho}{\partial\theta} + \rho\left\{\left(\frac{\partial u_x}{\partial x}\right)_{yz\theta} + \left(\frac{\partial u_y}{\partial y}\right)_{x,z,\theta} + \left(\frac{\partial u_z}{\partial z}\right)_{x,y,\theta}\right\}\right) + \rho\frac{du_x}{d\theta}\right] dx\,dy\,dz \qquad 5\text{-}38$$

The portion of the integrand between parentheses ( ) is zero since it is equivalent to Equation (5-3), the Equation of Continuity. Therefore

$$\frac{d}{d\theta}\iiint \rho u_x \, dx\,dy\,dz = \iiint \rho\frac{du_x}{d\theta}\, dx\,dy\,dz \qquad 5\text{-}39$$

(Note that this result would also have been obtained for any other continuous variable than u_x. Thus if ξ_1 is any function of x, y, z, and θ which satisfies the same conditions as ξ above,

$$\frac{d}{d\theta}\iiint \rho\xi_1 \, dx\,dy\,dz = \iiint \rho\frac{d\xi}{d\theta}\, dx\,dy\,dz). \qquad 5\text{-}40$$

Returning to Equation (5-31), substitute the result obtained in Equation (5-39) in it, and transform the surface integral by Gauss' theorem, Equation (5-5), giving, 5-41

$$\iiint\left[\rho\frac{du_x}{d\theta} - \Phi_x\rho - \left(\frac{\partial\tau_{xx}}{\partial x}\right)_{y,z,\theta} - \left(\frac{\partial\tau_{yx}}{\partial y}\right)_{x,z,\theta} - \left(\frac{\partial\tau_{zx}}{\partial z}\right)_{x,y,\theta}\right] dx\,dy\,dz = 0$$

Since the region R throughout which the integration is taken is arbitrary, the integrand must be identically zero, giving Equation (5-28).

Equations (5-29) and (5-30) are obtained in an exactly similar manner.

Not all of the surface stresses given Equations (5-28), (5-29), and (5-30) and shown in Figure 5-6 are independent, however. To find their interrelations, consider the moments of all the forces acting on the volume element of Figure 5-6 about axes through the geometrical center of the element and parallel to each of the coordinate axes. Since the fluid properties and in particular the density are assumed to be continuous functions, they may be expressed as linear functions of the dimensions of the volume element. Therefore, the center of gravity and the center of rotation differ from the geometrical center at most only by distances of the order of magnitude of the square of the dimensions of the volume element, i.e. by negligible amounts compared to the dimensions of the element.

The normal stresses τ_{xx}, τ_{yy}, and τ_{zz} act along lines passing through the geometrical center of the element, and the resultants of the external forces act along lines passing through the center of gravity. Therefore, these forces exert moments which are at least a higher order of smallness compared to those exerted by the surface shear stresses τ_{xy}, etc., and hence they can be neglected in comparison.

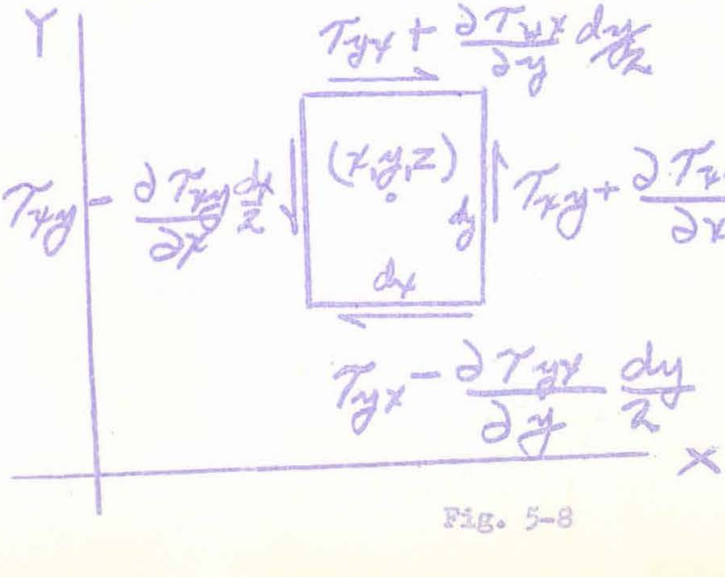

Fig. 5-8

In Figure 5-8 is shown a section of the volume element of Figure 5-6 through the point (x, y, z) and perpendicular to the Z-axis.

The moment exerted by the shear stresses which are shown must equal the product of the moment of inertia of the element about the axis through the point (x, y, z) and parallel to the Z-axis by its angular acceleration about the same point, since the moments of the other forces acting in this plane are of a higher order of smallness and may be neglected.

Therefore

$$\left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\cdot\frac{dx}{2}\right) dy\,dz\,\frac{dx}{2} + \left(\tau_{xy} - \frac{\partial \tau_{xy}}{\partial x}\cdot\frac{dx}{2}\right) dy\,dz\,\frac{dx}{2}$$

$$- \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\cdot\frac{dy}{2}\right) dx\,dz\,\frac{dy}{2} - \left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\cdot\frac{dy}{2}\right) dx\,dz\,\frac{dy}{2}$$

$$\cong \rho\,dx\,dy\,dz\left(\frac{dx^2+dy^2}{12}\right)\alpha \qquad 5\text{-}42$$

where α is the angular acceleration.[1] Simplifying

$$\tau_{xy} - \tau_{yx} \cong \rho\left(\frac{dx^2+dy^2}{12}\right)\alpha \qquad 5\text{-}43$$

Since the dimensions of the volume element may be made as small as desired, in the limit, at the point (x, y, z),

$$\tau_{xy} = \tau_{yx} \qquad 5\text{-}44$$

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1 Timoshenka, S. and Young, D. H., Engineering Mechanics, McGraw Hill Book Company (1937).

and similarily

$$\tau_{xz} = \tau_{zx} \qquad 5\text{-}45$$

$$\tau_{yz} = \tau_{zy} \qquad 5\text{-}46$$

5-7 Acceleration of the Flowing Fluid (continued)

In order to utilize the momentum Equations (5-28), (5-29), and (5-30) it is necessary to evaluate the surface stresses used in them in terms of known or calculable quantities. To achieve this purpose, it is necessary to examine the motion of an element of volume of the flowing fluid more closely.

It has been proved[1] that the motion of an element of volume of a flowing fluid may be analyzed into three independent motions: a pure translation of the element as a whole, a pure rotation of the element as a whole about an instantaneous center, and a pure distortion or strain of the element.

Assuming the result, it is possible to determine the magnitude of each of these types of motion by considering the behavior of an element of volume of the fluid which has edges of length dx, dy, and dz parallel to the respective coordinate axes at time θ. See Figure 5-9.

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1 See Page or Milne-Thomson.

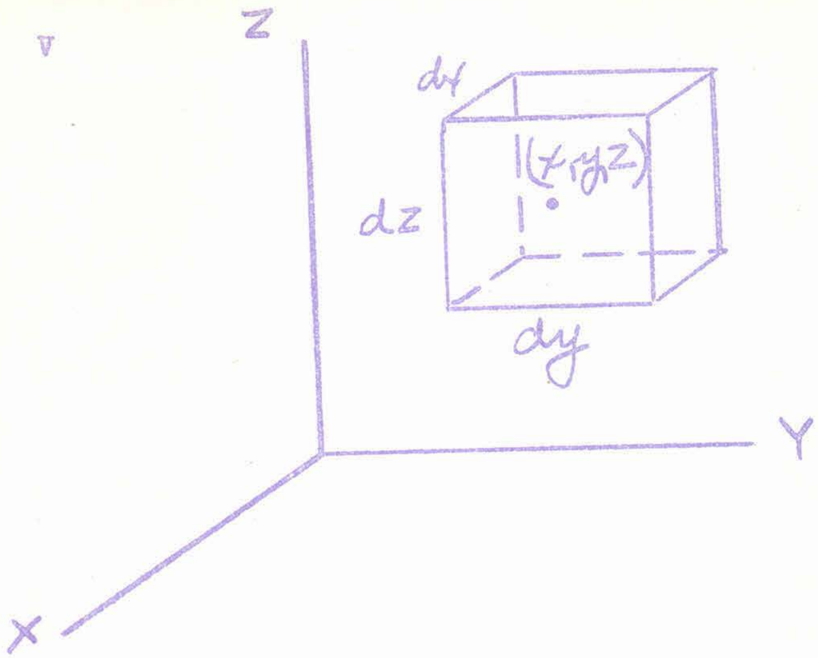

Fig. 5-9

If the components of the fluid velocity at the corner point (x, y, z) nearest the origin are u_x, u_y, and u_z, the components of the velocity in the X, Y, and Z-directions at every other corner point will differ from these by amounts depending on the lengths of the edges and the gradients of the velocity in the relevant directions.

In order to avoid possible confusion, only the motion of one face of the volume element will be considered; the results being easily extensible to the motions of the other faces; and, therefore, to the motion of the element itself. Take, for example, the face nearest the X-Y plane.

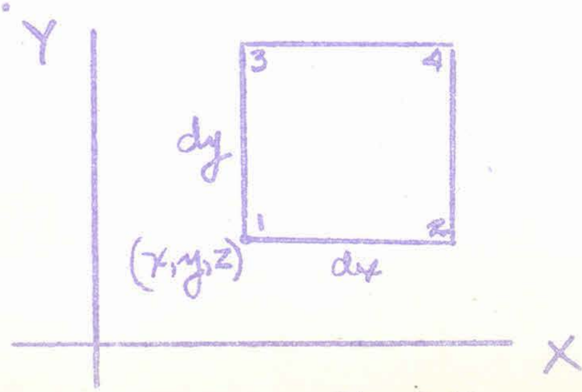

Fig. 5-10

The velocity components in the X and Y directions at the various corners at time θ will be, therefore:

Corner Point/Direction	X	Y
1	u_x	u_y
2	$u_x + \frac{\partial u_x}{\partial x} dx$	$u_y + \frac{\partial u_y}{\partial x} dx$
3	$u_x + \frac{\partial u_x}{\partial y} dy$	$u_y + \frac{\partial u_y}{\partial y} dy$
4	$u_x + \frac{\partial u_x}{\partial y} dy + \frac{\partial u_x}{\partial x} dx$	$u_y + \frac{\partial u_y}{\partial x} dx + \frac{\partial u_y}{\partial y} dy$

At time $\theta + d\theta$, the face will have become, say,

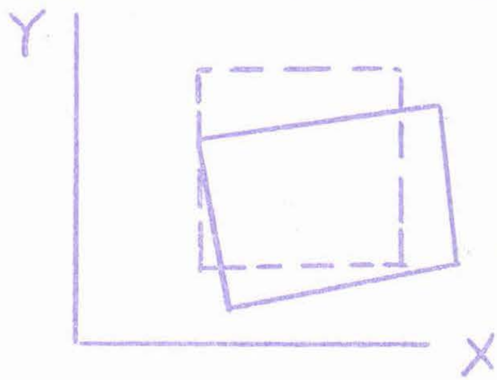

Fig. 5-11

For pictorial purposes the distortion has been drawn as finite, so that the motion of the face cannot be exactly analyzed into a motion of pure translation, of pure rotation, and of pure distortion.

Approximately, however, the finite motion may be analyzed into a motion of pure translation:

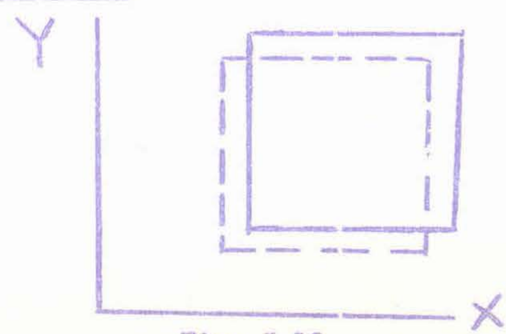

Fig. 5-12

In the time increment $d\theta$, the magnitude of the translation of the face of the element in the X-direction is $u_x d\theta$, the rate of translation in the X- direction is u_x, and the acceleration of the linear translation in the X-direction is $\left(\frac{\partial u_x}{\partial \theta}\right)_{x,y,z}$. Similar results hold for the Y-direction and for the Z-direction in other faces.

A motion of pure rotation:

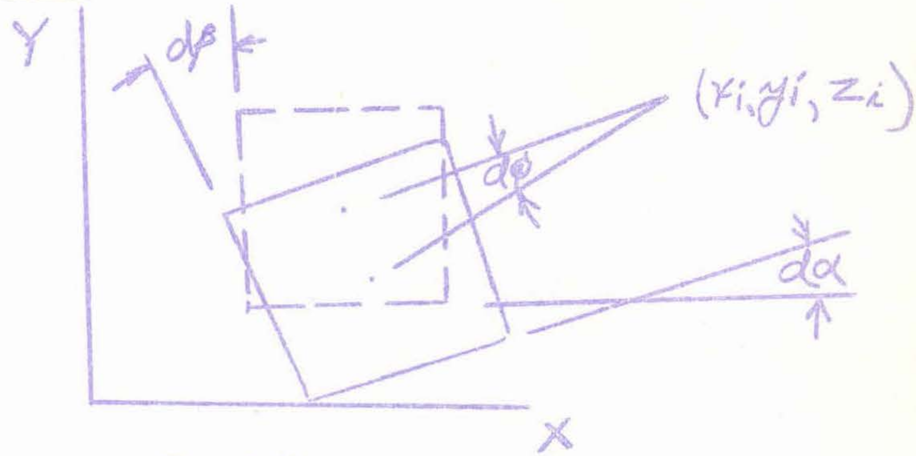

Fig. 5-13

The motion of pure rotation is about an axis parallel to the Z-axis and passing through the instantaneous center (x_i, y_i, z_i). It can be measured by the angle $d\phi$ through which the line connecting the axis and the center of the element rotate in the time $d\theta$. It may also be measured by the angle through which the edges of this face rotate in the same time interval; however, the latter are rotating at different rates, so that the average of the angles through which two adjacent edges rotate is a better measure of $d\phi$. Taking the edges nearest the axes (the others differ in velocity from these only by infinitesimals) and remembering that for infinitesimal angles, the angle and its tangent are equal,

$$d\alpha = \frac{\left(\frac{\partial u_y}{\partial x}\right) dx\, d\theta}{dx} \;\&\; d\beta = -\left(\frac{\partial u_x}{\partial y}\right) dy\, d\theta$$

since the gradients were assumed positive (see table on p-30) so that

$$d\phi = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) d\theta$$

The rate of rotation or angular velocity about (x_i, y_i, z_i) is

$$\frac{1}{2}\left[\left(\frac{\partial u_y}{\partial x}\right)_{y z \theta} - \left(\frac{\partial u_x}{\partial y}\right)_{x, z, \theta}\right] = \omega_z \qquad 5\text{-}47$$

where ω_z is the Z-component of the angular velocity. (The Z-component because the rotation is about an axis parallel to the Z-axis). The signs of the terms in Equation (5-47) may be shown to be correct for rotation about any other instantaneous center if it is remembered that counterclockwise rotation is defined as positive.

From a similar analysis of the motion of the other faces of the element of Figure 5-9, the other components of the angular velocity are found to be

$$\omega_y = \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial z}\right)_{x, y, \theta} - \left(\frac{\partial u_z}{\partial x}\right)_{y, z, \theta}\right] \qquad 5\text{-}48$$

$$\omega_x = \frac{1}{2}\left[\left(\frac{\partial u_z}{\partial y}\right)_{x,z,\theta} - \left(\frac{\partial u_y}{\partial z}\right)_{x,y,\theta}\right] \quad 5\text{-}49$$

A motion of pure distortion:

The motion of pure distortion may be further analyzed into a motion of pure linear distortion or extension, and a motion of pure angular distortion. Taking the second of these:

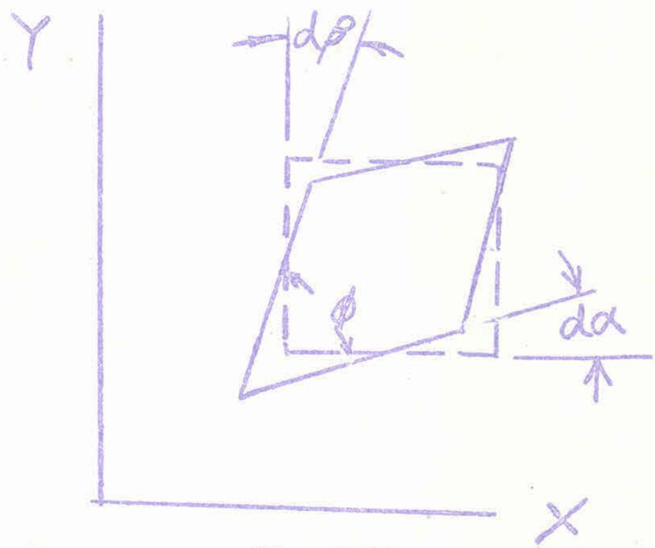

Fig. 5-14

The pure angular deformation of the face of the element (see Figure 5-14) may be measured by the changes in the corner angles in the time $d\theta$. Since the lengths of the edges are unchanged, the changes in all the angles are the same, except for signs. Taking the corner nearest the origin (the conditions at the other corners differ only by infinitesimals); and since the angular deformation is the difference between the angles through which the adjacent sides have rotated in the time $d\theta$,

$$-d\phi = d\alpha - (-d\beta)$$
$$= \frac{\frac{\partial u_y}{\partial x} dx\, d\theta}{dx} - \left(- \frac{\frac{\partial u_x}{\partial y} dy\, d\theta}{dy} \right)$$
$$= \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right) d\theta$$

again remembering that counterclockwise rotation is taken as positive, and the angle and its tangent are equal for infinitesimal angles. Thus the rate of angular deformation is

$$2\,\zeta_z = \left[\left(\frac{\partial u_y}{\partial x} \right)_{y,z,\theta} + \left(\frac{\partial u_x}{\partial y} \right)_{x,z,\theta} \right] \quad 5\text{-}50$$

and it is defined as 2ζ because the distortion is for the face normal to the Z-axis, and because the symmetry of some equation soon to be derived is improved by introducing the arbitrary factor 2 .

The rates of angular distortion for the faces normal to the Y and X-axis respectively, are, by a similar analysis:

$$2\,\zeta_y = \left[\left(\frac{\partial u_x}{\partial z} \right)_{x,y,\theta} + \left(\frac{\partial u_z}{\partial x} \right)_{y,z,\theta} \right] \quad 5\text{-}51$$

$$2\,\zeta_x = \left[\left(\frac{\partial u_z}{\partial y} \right)_{x,z,\theta} + \left(\frac{\partial u_y}{\partial z} \right)_{x,y,\theta} \right] \quad 5\text{-}52$$

completing the analysis of the angular distortion of the element.

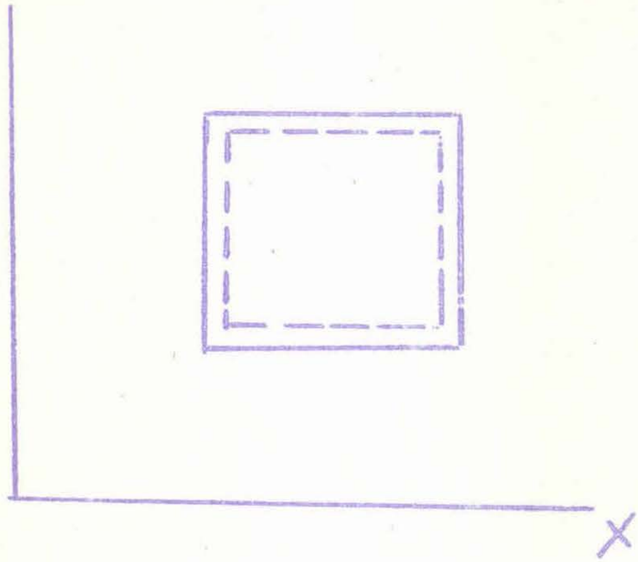

Fig. 5-15

The pure linear distortion of the face of the element (See Figure 5-15) may be most profitably analyzed by further decomposing it into a pure linear distortion without change of volume, and a pure linear distortion in which the lengths of the edges are altered proportionally (ie. a pure expansion or compression). The linear distortion of the first sort (See Figure (5-16) may be measured

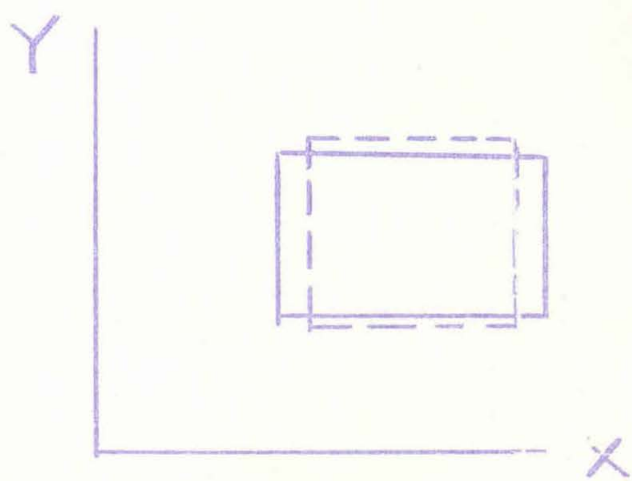

Fig. 5-16

by the relative displacement of opposite edges during the time interval $d\theta$. Thus for the X-direction, the linear distortion is $\left[\left(u_x' + \frac{\partial u_x'}{\partial x}dx\right) - u_x'\right]d\theta$ $= \frac{\partial u_x'}{\partial x}dx\,d\theta$, the rate of linear deformation is $\frac{\partial u_x'}{\partial x}dx$, and the acceleration in the X-direction is $u_x \frac{\partial u_x'}{\partial x}$, where u_x' is the velocity of the edge nearest the Y-axis and parallel to it, after the velocity due to the second type of distortion is allowed for.

By drawing segments connecting consecutive bisectors of the edges of the face of the element (See Figure 5-17), it may

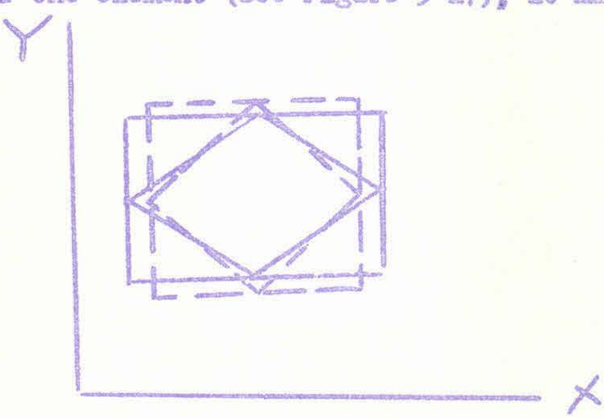

Fig. 5-17

be seen that this type of pure linear distortion contains elements of pure angular distortion within it. A similar construction in Figure 5-14 shows that pure angular distortion also contains elements of pure linear distortion of this type, both of which interrelations may be emphasized by a rotation of the coordinate axes through 45°.

The second type of pure linear distortion is closely related to the divergence of the velocity, since as was shown in section 1, this quantity gives the relative volume rate of expansion of the fluid. Since the increases in the lengths of the edges were assumed proportional to those lengths, the change in the volume of the fluid element is proportional to the cube of any edge, dx, say; so that the relative rate of expansion of the edge dx, $\frac{1}{dx}\frac{\partial dx}{\partial \theta}$ is equal to 1/3 the divergence of the velocity for this type of distortion, as may be seen by substituting ∂dx^3 for V. (The total relative rate of expansion of the edge dx is $\frac{\partial u_x}{\partial x}$). The rate of expansion is thus given by $\frac{dx}{3}$ (div u), and the acceleration for this type of distortion in the X-direction by $\frac{u_x}{3}$ div u.

If no such decomposition of the pure linear distortion of the face of the element is made (See Figure 5-15), a measure of this distortion in the X-direction is given by $(u_x + \frac{\partial u_x}{\partial x})d\theta - u_x d\theta = \frac{\partial u_x}{\partial x} dx\, d\theta$, of the rate by $\frac{\partial u_x}{\partial x} dx$, and of the acceleration in the X-direction by $u_x \frac{\partial u_x}{\partial x}$.

The various relations which have been developed to express the motion of an element of volume of a flowing fluid may be related to the expressions which were derived to express the acceleration of a fluid (Equations (5-18), 5-19), and (5-20)) in the following manner.

If $\frac{1}{2}u_y\frac{\partial u_y}{\partial x} + \frac{1}{2}u_z\frac{\partial u_z}{\partial z}$ be added and subtracted from Equation (5-18), the result may be rearranged to give

$$a_x = \frac{\partial u_x}{\partial \theta} + u_x\frac{\partial u_x}{\partial x} + \frac{1}{2}u_y\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) - \frac{1}{2}u_y\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)$$
$$+ \frac{1}{2}u_z\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) + \frac{1}{2}u_z\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) \qquad 5\text{-}53$$

On substituting Equations (5-50), (5-51), (5-47), and (5-48) in Equation (5-53)

$$a_x = \left(\frac{\partial u_x}{\partial \theta}\right)_{x,y,z} + u_x\left(\frac{\partial u_x}{\partial x}\right)_{y,z,\theta} + u_y \zeta_z + u_z \zeta_y - u_y \omega_z + u_z \omega_y \qquad 5\text{-}54$$

and by a similar procedure Equations (5-19) and (5-20) become

$$a_y = \left(\frac{\partial u_y}{\partial \theta}\right)_{x,y,z} + u_y\left(\frac{\partial u_y}{\partial y}\right)_{x,z,\theta} + u_z \zeta_x + u_x \zeta_z - u_z\omega_x + u_x\omega_z \qquad 5\text{-}55$$

$$a_z = \left(\frac{\partial u_z}{\partial \theta}\right)_{x,y,z} + u_z\left(\frac{\partial u_z}{\partial z}\right)_{x,y,\theta} + u_x \zeta_y + u_y \zeta_x - u_x\omega_y + u_y\omega_x \qquad 5\text{-}56$$

The symmetrical form of these equations is the justification for the arbitrary factor 2 in Equations (5-50), (5-51), and (5-52).

5-8 Evaluation of the Surface Stresses in a Flowing Fluid

The equations which have been derived thus far are quite general, and they may be applied to both Newtonian and non-Newtonian fluids as well as to elastic and plastic solids. (See Chapter I Sec. 5 for the definitions of these types of materials.). This discussion is limited to the case of Newtonian fluids, but the analysis has been extended to the more general cases[1], which are of great technical importance.

The analysis of the motion of a fluid element of volume given in the last section shows that this motion may be separated into a translation of the element as a whole, a rotation of the element as a whole, and a distortion of the element. The definition of a Newtonian fluid given in Chapter I Section 5 and Equation (1-2) indicates that any motion of the fluid element as a whole,

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1 See W. Prager Theory of Plasticity Brown University (1941).
The Chemistry of Large Molecules Interscience Publ. Company, (1943).
A.E.H. Love, Mathematical Theory of Elasticity Cambridge University Press.

ie. as a rigid body, would not produce any viscous stresses, and that the deformation of the element would be the only source of such stresses.

In order to evaluate the surface stresses used in the Momentum Equations(5-28), (5-29), and (5-30) in terms of known or calculable quantities, it is necessary to construct a generalization of Equation (1-2) which retains the basic assumption for Newtonian fluids that the surface shear stresses are a linear function of the rates of strain, and also satisfies the relations which have been derived. (This is the point at which the analysis of the motion of an elastic solid, a plastic solid, a non-Newtonian fluid and a Newtonian fluid diverge; since for the first the stresses are determined by the strains, for the second by both the strains and the rates of strain, and for the third the stressesare a more general function of the rates of strain than a simple linear combination.) A rigorous derivation of the generalization is rather difficult, and it can best be given only by use of the tensor calculus[1].

However, the generalization can be made plausible with the help of a few of the results of the more rigorous analysis.

Thus a comparison of Figure 5-8 showing the surface shear stresses acting on the faces of the element of volume parallel to the Z-axis with Figure 5-14 showing the pure angular distortion of the face normal to the Z-axis makes it plausible that there is a cause and effect relation between them, ie. that

$$\tau_{xy} = \tau_{yx} = \eta\left[\left(\frac{\partial u_y}{\partial x}\right)_{y,z,\theta} + \left(\frac{\partial u_x}{\partial y}\right)_{x,z,\theta}\right] \qquad 5\text{-}57$$

- -

1 See Lamb, Milne-Thomson, Page, or McConnell Applications of the Absolute Differential Calculus. Blackie and Son.

and similarily for the other coordinate directions

$$\tau_{xz} = \tau_{zx} = \eta\left[\left(\frac{\partial u_x}{\partial z}\right)_{x,y,\theta} + \left(\frac{\partial u_z}{\partial x}\right)_{y,z,\theta}\right] \quad 5\text{-}58$$

$$\tau_{yz} = \tau_{zy} = \eta\left[\left(\frac{\partial u_y}{\partial z}\right)_{x,y,\theta} + \left(\frac{\partial u_z}{\partial y}\right)_{x,z,\theta}\right] \quad 5\text{-}59$$

where η is the absolute viscosity. (Compare Equation (1-2)).

The evaluation of the normal stresses τ_{xx}, τ_{yy}, and τ_{zz} is more difficult. It has been shown[1] that whatever orientation of the coordinate axes is taken, the sum of the three normal stresses is constant ie. an invariant. Thus if τ_{ave} is taken as the average normal stress,

$$\tau_{ave} = \frac{1}{3}\left(\tau_{xx} + \tau_{yy} + \tau_{zz}\right) \quad 5\text{-}60$$

It is not hard to show[2] that for a perfect fluid, ie. a fluid without viscosity, that

$$\tau_{ave} = \tau_{xx} = \tau_{yy} = \tau_{zz} = -P \quad [o] \quad 5\text{-}61$$

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1 See footnote on previous page.
2 See Lamb p-1.

where P is the thermodynamic pressure[1]. The minus sign occurs because in order to make the Momentum Equations more symmetrical, the normal stresses were taken as tensions, whereas P is a pressure. (See Figure 5-6).

It is plausible, therefore, to identify τ_{ave} with the normal surface stresses which produce pure expansion or compression of the fluid, that is, pure linear distortion of the second type, and to assume that the residual normal stresses $\tau_{xx}-\tau_{ave}$, $\tau_{yy}-\tau_{ave}$ and, $\tau_{zz}-\tau_{ave}$ are the result of the pure linear distortion of the first type which involves deformation of the element of volume of the fluid without change of volume and which is similar to pure angular deformation.

Just as a rotation of the coordinate axes through 45° reveals interrelations between angular deformation and pure linear deformation of the first type, the same rotation partially interconverts shear and normal stresses (See Figure 5-6). Therefore the explicit expression for the residual normal stresses must contain a term similar to those for the shear stresses as in Equations (5-57), (5-58), and (5-59). The exact analysis indicates that these stresses should also be a general linear function of the relative rates of expansion of the edges of the element of volume, or, for , for example,

$$\tau_{xx}=\tau_{ave}+\eta\left[\left(\frac{\partial u_x}{\partial x}\right)+\left(\frac{\partial u_x}{\partial x}\right)\right]+\alpha\left(\frac{\partial u_x}{\partial x}\right)+\beta\left(\frac{\partial u_y}{\partial y}\right)+\gamma\left(\frac{\partial u_z}{\partial z}\right) \qquad 5\text{-}62$$

where α, β, and γ are constants (or more precisely, functions of state).

1 See Section 1-4.

However, it has been implicity assumed that the fluid is isotropic, that is, has the same properties in all directions; so since the linear function is the same for each stress, while a direct transformation of Equation (5-62) from τ_{xx} say to another stress (or a rotation of the axes through 90°) gives a different function unless

$$\alpha = \beta = \gamma \qquad 5\text{-}63$$

this relation must be true. This type of analysis is usually shortened to the statement: Since the fluid is isotropic, by symmetry $\alpha = \beta = \gamma$.

(Non-isotropic or anisotropic fluids are relatively rare, being confined to the class of liquids known as liquid crystals. Many colloidal solutions, however, are anisotropic.)

Therefore, the explicit expressions for the normal surface stresses are:

$$\tau_{xx} = \tau_{ave} + 2\gamma\left(\frac{\partial u_x}{\partial x}\right)_{yz\theta} + \alpha\left[\left(\frac{\partial u_x}{\partial x}\right)_{y,z\theta} + \left(\frac{\partial u_y}{\partial y}\right)_{x,z,\theta} + \left(\frac{\partial u_z}{\partial z}\right)_{xy\theta}\right] \qquad 5\text{-}64$$

$$\tau_{yy} = \tau_{ave} + 2\gamma\left(\frac{\partial u_y}{\partial y}\right)_{x,z,\theta} + \alpha\left[\left(\frac{\partial u_x}{\partial x}\right)_{yz\theta} + \left(\frac{\partial u_y}{\partial y}\right)_{x,z,\theta} + \left(\frac{\partial u_z}{\partial z}\right)_{xy\theta}\right] \qquad 5\text{-}65$$

$$\tau_{zz} = \tau_{ave} + 2\gamma\left(\frac{\partial u_x}{\partial z}\right)_{xy\theta} + \alpha\left[\left(\frac{\partial u_x}{\partial x}\right)_{y,z,\theta} + \left(\frac{\partial u_y}{\partial y}\right)_{x,z,\theta} + \left(\frac{\partial u_z}{\partial z}\right)_{xy\theta}\right] \qquad 5\text{-}66$$

Adding these equations,

$$\tau_{xx} + \tau_{yy} + \tau_{zz} = 3\,\tau_{ave} + (2\eta + 3\alpha)\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \qquad 5\text{-}67$$

Therefore, in general, from Equation (5-60)

$$\alpha = -\frac{2}{3}\eta \qquad 5\text{-}68$$

and α can be eliminated among Equations (5-64), (5-65), (5-66) and (5-68).

Equations(5-57), (5-58), (5-59), (5-64), 5-65), (5-66), and (5-68) are the desired generalization of Equation (1-2) for the general case of the flow of a Newtonian fluid, with the exception that the average normal stress τ_{ave} has not been evaluated.

It has been shown by Chapman and Cowling[1] from the kinetic theory of gases alone that

$$p = -\tau_{ave} \qquad [0] \quad 5\text{-}69$$

and that the surface stresses are given by the above mentioned equations to a very good approximation for perfect gas composed of rigid smooth spheres. However, an even better approximation was obtained by the same authors for this case which gave Equation (5-69) again, but more complicated expressions for the surface stresses involving terms, which are usually very small, depending on temperature and pressure gradients as well as velocity gradients as in the above equations.

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1 Chapman and Cowling, The Mathematical Theory of Non-Uniform Gases. MacMillan (1939).

For real gases, however, especially near the critical state, it is very likely that the average normal stress, τ_{ave}, depends on the rate of expansion of the gas, since the intermolecular forces exert attractions among the molecules of the gas which would delay their free expansion and hence diminish the normal stress which they collectively could exert. As an approximation to the behavior of highly compressed gases,[1] Equation (5-69) may be modified by the inclusion of a term on the right-hand side of the form $\eta' \operatorname{div} u$, where η' is a second coefficient of viscosity, since div u represents the rate of expansion of the gas. Under these conditions, however, the applicability of the concept of thermodynamic pressure is beginning to be questionable so that even this assumption may be incorrect in some situations.

For a liquid, since the rate of expansion can only be very small because large pressure or temperature variations produce but small density changes (Equation 5-10), it seems reasonable to assume that Equation (5-69) applies. The lack of a satisfactory molecular theory of the liquid state prevents any more accurate approximations.

It may also be necessary to modify the equations for the surface stresses to include terms similar to those obtained by Chapman and Cowling to allow for the effects of pressure and especially temperature variations.

However, the above critical comments must not be taken to mean that the equations under discussion are only very crude approximations; on the contrary, they have been shown experimentally to be very excellent approximations (i.e. within the experimental error) to the actual behavior of real gases and liquids. Further comments on this subject will be found in Chapter IX.

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1 Lamb p-645.

5-9 Stokes-Navier Equations

Thus the general equations expressing the conservation of momentum for a Newtonian fluid which behaves volumetrically either like a perfect gas or a perfect liquid, at least approximately, are, from Equations (5-28),(5-29), (5-30), 5-57), (5-58), (5-59), (5-64), (5-65), (5-66),(5-68), and (5-69),

$$\rho a_x = \rho \Phi_x - \frac{\partial P}{\partial x} + \frac{1}{3}\frac{\partial}{\partial x}\eta\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + \frac{\partial}{\partial x}\left(\eta\frac{\partial u_x}{\partial x}\right) + \frac{\partial}{\partial y}\left(\eta\frac{\partial u_x}{\partial y}\right) + \frac{\partial}{\partial z}\left(\eta\frac{\partial u_x}{\partial z}\right) \qquad (n)\ 5\text{-}70$$

$$\rho a_y = \rho \Phi_y - \frac{\partial P}{\partial y} + \frac{1}{3}\frac{\partial}{\partial y}\eta\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + \frac{\partial}{\partial x}\left(\eta\frac{\partial u}{\partial}\right. + \frac{\partial}{\partial y}\left(\eta\frac{\partial u_y}{\partial y}\right) + \frac{\partial}{\partial z}\left(\eta\frac{\partial u_y}{\partial z}\right) \qquad (n)\ 5\text{-}71$$

$$\rho a_z = \rho \Phi_z - \frac{\partial P}{\partial z} + \frac{1}{3}\frac{\partial}{\partial z}\eta\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + \frac{\partial}{\partial x}\left(\eta\frac{\partial u}{\partial}\right. + \frac{\partial}{\partial y}\left(\eta\frac{\partial u_z}{\partial y}\right) + \frac{\partial}{\partial z}\left(\eta\frac{\partial u_z}{\partial z}\right) \qquad (n)\ 5\text{-}72$$

These equations are known as the Stokes-Navier Equations in honor of some of the men who first derived them.

(Note: For simplicity, the subscripts on the partial derivatives have been omitted on these final equations as has often been the case in derivations, but they must always be understood.)

In the usual case where gravity is the only important external force, the first two terms on the right side of Equations (5-70), (5-71), and (5-72) may be combined to give $-\frac{\partial}{\partial x}(P+\sigma h)$, $-\frac{\partial}{\partial y}(P+\sigma h)$, & $-\frac{\partial}{\partial z}(P+\sigma h)$ respectively, where $\sigma = g\rho$, <u>if</u> the fluid is incompressible, when the third term on the right side disappears; (See Equation 5-3) so that, for a perfect liquid, the Stokes-Navier Equations become:

$$\rho a_x = -\frac{\partial}{\partial x}(P+\sigma h) + \frac{\partial}{\partial x}\left(\eta\frac{\partial u_x}{\partial x}\right) + \frac{\partial}{\partial y}\left(\eta\frac{\partial u_x}{\partial y}\right) + \frac{\partial}{\partial z}\left(\eta\frac{\partial u_x}{\partial z}\right) \quad [0,m,l] \quad 5\text{-}73$$

$$\rho a_y = -\frac{\partial}{\partial y}(P+\sigma h) + \frac{\partial}{\partial x}\left(\eta\frac{\partial u_y}{\partial x}\right) + \frac{\partial}{\partial y}\left(\eta\frac{\partial u_y}{\partial y}\right) + \frac{\partial}{\partial z}\left(\eta\frac{\partial u_y}{\partial z}\right) \quad [0,m,l] \quad 5\text{-}74$$

$$\rho a_z = -\frac{\partial}{\partial z}(P+\sigma h) + \frac{\partial}{\partial x}\left(\eta\frac{\partial u_z}{\partial x}\right) + \frac{\partial}{\partial y}\left(\eta\frac{\partial u_z}{\partial y}\right) + \frac{\partial}{\partial z}\left(\eta\frac{\partial u_z}{\partial z}\right) \quad [0,m,l] \quad 5\text{-}75$$

For the case of objects moving through a large volume of fluid, the results obtained by assuming that the viscosity is zero in the Stokes-Navier Equations (when the last four terms on the right sides of Equations (5-70), (5-71), and (5-72) disappear), are remarkably accurate for all portions of the flow, except in the immediate vicinity of the obstacle, provided that no velocities comparable to that of sound are involved. Many solutions of the equations under this restriction are discussed in texts on hydrodynamics[1] and

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1 See Lamb or Milne-Thomson.

aerodynamics. Such problems, are, however, usually of little importance in chemical engineering.

5-10 Stokes-Navier Equations in Cylindrical Coordinates

In distinction to the situation which exists for ordinary differential equations, the task of bringing the solution of a partial differential equation into agreement with the boundary conditions is usually of the same order of difficulty as that of solving the equation originally. Consequently any simplification in the boundary conditions is of great convenience.

In case the boundary to the flow is a cylinder, for example; if cylindrical instead of Cartesian coordinates are used the boundary conditions are to be satisfied at $r = r_o$ instead of $(x^2 + y^2)^{\frac{1}{2}} = r_o$, where r_o is the radius of the cylinder. This apparently slight simplification is often of great practical importance so the Stokes-Navier Equations will be given in cylindrical coordinates together with the accelerations and the angular velocities. (The latter quantities will be discussed in sec. 5-).

$$\rho \frac{du_r}{d\theta} = \rho \Phi_r - \frac{\partial p}{\partial r} + \frac{1}{3}\frac{\partial}{\partial r}\left[\frac{\eta}{r}\left(\frac{\partial r u_r}{\partial r} + \frac{\partial u_\psi}{\partial \psi} + \frac{\partial r u_x}{\partial x} \right.\right.$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\eta r \frac{\partial u_r}{\partial r} \right) + \frac{1}{r}\frac{\partial}{\partial \psi}\left(\eta \frac{\partial u_r}{\partial x} \right) + r\frac{\partial}{\partial x}\left(\eta \frac{\partial u_r}{\partial x} \right) \right] \qquad [n] \quad 5\text{-}7$$

$$\rho \frac{du_x}{d\theta} = \rho \Phi_\psi - \frac{1}{r}\frac{\partial p}{\partial \psi} + \frac{1}{3r}\frac{\partial}{\partial \psi}\left[\frac{\eta}{r}\left(\frac{\partial r u_r}{\partial r} + \frac{\partial u_\psi}{\partial \psi} + \frac{\partial r u_x}{\partial x} \right) \right]$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\eta r \frac{\partial u_\psi}{\partial r} \right) + \frac{1}{r}\frac{\partial}{\partial x}\left(\eta \frac{\partial u_\psi}{\partial \psi} \right) + r\frac{\partial}{\partial x}\left(\eta \frac{\partial u_x}{\partial x} \right) \right] \qquad [n] \quad 5\text{-}7$$

$$\rho \frac{du_x}{d\theta} = \rho \Phi_x - \frac{\partial P}{\partial x} + \frac{1}{3}\frac{\partial}{\partial x}\left[\eta\left(\frac{\partial r u_r}{\partial r} + \frac{\partial u_\psi}{\partial \psi} + \frac{\partial r u_x}{\partial x}\right)\right]$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\eta r \frac{\partial u_x}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial x}\left(\eta \frac{\partial u_x}{\partial x}\right) + r\frac{\partial}{\partial x}\left(\eta \frac{\partial u_x}{\partial x}\right)\right] \quad 5\text{-}78$$

where

$$\frac{du_r}{d\theta} = \frac{\partial u_r}{\partial \theta} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\psi}{r}\frac{\partial u_r}{\partial \psi} + u_x \frac{\partial u_r}{\partial x} - \frac{u_\psi^2}{r} \quad 5\text{-}79$$

$$\frac{du_\psi}{d\theta} = \frac{\partial u_\psi}{\partial \theta} + u_r \frac{\partial u_\psi}{\partial r} + \frac{u_\psi}{r}\frac{\partial u_\psi}{\partial \psi} + u_x \frac{\partial u_\psi}{\partial x} + \frac{u_r u_\psi}{r} \quad 5\text{-}80$$

$$\frac{du_x}{d\theta} = \frac{\partial u_x}{\partial \theta} + u_r \frac{\partial u_x}{\partial r} + \frac{u_\psi \partial u_x}{\partial \psi} + u_x \frac{\partial u_x}{\partial x} \quad 5\text{-}81$$

and

$$\omega_r = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_x}{\partial u_\psi} - \frac{\partial u_\psi}{\partial x}\right) \quad 5\text{-}82$$

$$\omega_\psi = \frac{1}{2}\left(\frac{\partial u_r}{\partial x} - \frac{\partial u_x}{\partial r}\right) \quad 5\text{-}83$$

$$\omega_x = \frac{1}{2} \cdot \frac{1}{r}\left(\frac{\partial r u_\psi}{\partial r} - \frac{\partial u_r}{\partial \psi} \right) \qquad 5\text{-}84$$

5-11 Stokes-Navier Equations for Spherical Coordinates

In case the boundary to the flow is a sphere, equal simplification of the solution is usually obtained by solving in terms of spherical coordinates. The corresponding forms of the Stokes-Navier Equations in spherical coordinates are:

$$\rho \frac{du_r}{d\theta} = \rho \Phi_r - \frac{\partial P}{\partial r} + \frac{1}{3}\frac{\partial}{\partial r}\left[\frac{\eta}{r^2 \sin\phi}\left(\sin\phi \frac{\partial}{\partial r}(r u_r) + r\frac{\partial}{\partial \phi}(\sin\phi \, u_\phi) + r\frac{\partial u_\psi}{\partial \psi} \right)\right] + \frac{1}{r^2 \sin\phi}\left[\sin\phi \frac{\partial}{\partial r}\left(\eta r^2 \frac{\partial u_r}{\partial r}\right) + \frac{\partial}{\partial \phi}\left(\eta \sin\phi \frac{\partial u_r}{\partial \phi}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \psi}\left(\eta \frac{\partial u_r}{\partial \psi}\right)\right] \quad [11] \; 5\text{-}85$$

$$\rho \frac{du_\phi}{d\theta} = \rho \Phi_\phi - \frac{1}{r}\frac{\partial P}{\partial \phi} + \frac{1}{3r}\frac{\partial}{\partial \phi}\left[\frac{\eta}{r^2 \sin\phi}\left(\sin\phi \frac{\partial}{\partial r}(r^2 u_r) + r\frac{\partial}{\partial \phi}(\sin\phi \, u_\phi + r\frac{\partial u_\psi}{\partial \psi} \right)\right] + \frac{1}{r^2 \sin\phi}\left[\sin\phi \frac{\partial}{\partial r}\left(\eta r^2 \frac{\partial u_\phi}{\partial r}\right) + \frac{\partial}{\partial \phi}\left(\eta \sin\phi \frac{\partial u_\phi}{\partial \phi}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \psi}\left(\eta \frac{\partial u_\phi}{\partial \psi}\right)\right] \quad [11] \; 5\text{-}86$$

$$\rho \frac{du_\psi}{d\theta} = \rho \Phi_\psi - \frac{1}{r \sin\phi}\frac{\partial P}{\partial \psi} + \frac{1}{3 r \sin\phi}\frac{\partial}{\partial \psi}\left[\frac{\eta}{r^2 \sin\phi} \cdot \left(\sin\phi \frac{\partial (r^2 u_r)}{\partial r}\right) + r \frac{\partial}{\partial \phi}(\sin\phi\, u_\phi) + r\frac{\partial u_\psi}{\partial \psi}\right)\right] + \frac{1}{r^2 \sin\phi} \cdot$$

$$\left[\sin\phi \frac{\partial}{\partial r}\left(\eta r^2 \frac{\partial u_\psi}{\partial r}\right) + \frac{\partial}{\partial \phi}\left(\eta \sin\phi \frac{\partial u_\psi}{\partial \phi}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \psi}\left(\eta \frac{\partial u_\psi}{\partial \psi}\right)\right]$$

where [m] 5-87

$$\frac{du_r}{d\theta} = \frac{\partial u_r}{\partial \theta} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi} + \frac{u_\psi}{r \sin\phi}\frac{\partial u_r}{\partial \psi} - \frac{u_\phi^2 + u_\psi^2}{r}$$

5-88

$$\frac{du_\phi}{d\theta} = \frac{\partial u_\phi}{\partial \theta} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\psi}{r \sin\phi}\frac{\partial u_\phi}{\partial \psi} + \frac{u_r u_\phi}{r} - \frac{u_\psi^2 \cot\phi}{r}$$

and

5-89

$$\frac{du_\psi}{d\theta} = \frac{\partial u_\psi}{\partial \theta} + u_r \frac{\partial u_\psi}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_\psi}{\partial \phi} + \frac{u_\psi}{r \sin\phi}\frac{\partial u_\psi}{\partial \psi} + \frac{u_r u_\psi}{r} - \frac{u_\psi u_\phi \cot\phi}{r}$$

also

5-90

$$\omega_r = \frac{1}{2 r \sin\phi}\left[\frac{\partial}{\partial \phi}\left(\sin\phi\, u_\psi - \frac{\partial u_\phi}{\partial \psi}\right]\right. \qquad 5\text{-}91$$

$$\omega_\phi = \frac{1}{2r\sin\phi}\left[\frac{\partial u_r}{\partial \psi} - \frac{1}{2r}\frac{\partial (r u_\psi)}{\partial r}\right] \quad 5\text{-}92$$

$$\omega_\psi = \frac{1}{2r}\left[\frac{\partial}{\partial r}(r u_\phi) - \frac{\partial u_r}{\partial \phi}\right] \quad 5\text{-}93$$

5-12 Dimensionless Form of the Equations of Motion

For any particular flow situation, the Stokes-Navier Equations and the Equation of Continuity (Equations (5-70), (5-71), 5-72), and (5-3) respectively) may be reduced to a dimensionless form by substitution of "reduced" variables. The results which are obtained are important in two respects: First for the light they shed on the results of Dimensional Analysis, and second because the equations when written in this manner are in a very convenient form for numerical calculation since all the variables are reduced to a comparable basis.

It is usually possible to determine, at least approximately, the complete behavior of a particular flow situation by the specification of a few characteristic parameters of the flow. Thus if values of the pressure gradient $\left(\frac{\partial P}{\partial x}\right)_o$, density ρ_o, a characteristic length L_o, velocity U_o, and viscosity η_o which are in some fashion representative of the flow as a whole, are known, "reduced" dimensionless variables may be defined in the following way:

$$x' = x/L_0,\ y' = y/L_0,\ z' = z/L_0,\ u_x' = u_x/u_0,\ u_y' = u_y/u_0$$
$$u_z' = u_z/u_0,\ a_x' = a_x L_0/u_0^2,\ a_y' = a_y L_0/u_0^2,\ a_z' = a_z L_0/u_0^2$$
$$h' = h/L_0,\ \rho' = \rho/\rho_0,\ \eta' = \eta/\eta_0,\ \left(\frac{\partial P}{\partial x}\right)' = \frac{\partial P}{\partial x} \div \frac{\partial P}{\partial x}\Big|_0$$
$$\theta' = \theta u_0/L_0 \quad \Omega' = \Omega/u_0^2 = g h' L_0/u_0^2 \qquad 5\text{-}94$$

If the proper substitutions are made from Equations (5-94) in the Equation of Continuity, the simplified result is

$$\frac{\partial \rho'}{\partial \theta'} = -\rho'\left[\frac{\partial u_x'}{\partial x'} + \frac{\partial u_y'}{\partial y'} + \frac{\partial u_z'}{\partial z'}\right] \qquad 5\text{-}95$$

Similarily from Equation (5-70),

$$\frac{\rho_0 \rho' u_0^2}{L_0} a_x' = -\rho' \rho_0 g \frac{\partial h'}{\partial x'} - \left(\frac{\partial P}{\partial x}\right)_0 \left(\frac{\partial P}{\partial x}\right)' + \frac{1}{3}\frac{\eta_0 u_0}{L_0^2}\frac{\partial}{\partial x'}\left[\eta'\left(\frac{\partial u_x'}{\partial x'} + \frac{\partial u_y'}{\partial y'} + \frac{\partial u_z'}{\partial z'}\right)\right] + \frac{\eta_0 u_0}{L_0^2}\left[\frac{\partial}{\partial x'}\left(\eta'\left(\frac{\partial u_x'}{\partial x'}\right)\right) + \frac{\partial}{\partial y'}\left(\eta'\frac{\partial u_x'}{\partial y'}\right) + \frac{\partial}{\partial z'}\left(\eta'\frac{\partial u_x'}{\partial z'}\right)\right] \qquad 5\text{-}96$$

and similarly for the other two equations of the Stokes-Navier Equations.

Rearranging Equation (5-96)

$$\rho' a_x' = -\rho' g \frac{L_0}{u_0^2} \cdot \frac{\partial h'}{\partial x'} - \frac{L_0}{\rho_0 u_0^2}\left(\frac{\partial P}{\partial x}\right)_0 \left(\frac{\partial P}{\partial x}\right)' + \frac{1}{3}\frac{\eta_0}{L_0 u_0 \rho_0} \cdot \frac{\partial}{\partial x'}\left[\eta'\left(\frac{\partial u_x'}{\partial x'} + \frac{\partial u_y'}{\partial y'} + \frac{\partial u_z'}{\partial z'}\right)\right] + \frac{\eta_0}{L_0 u_0 \rho_0}\left[\frac{\partial}{\partial x'}\left(\eta'\frac{\partial u_x'}{\partial x'}\right) + \frac{\partial}{\partial y'}\left(\eta'\frac{\partial u_x'}{\partial y'}\right) + \frac{\partial}{\partial z'}\left(\eta'\frac{\partial u_x'}{\partial z'}\right)\right] \qquad 5\text{-}97$$

and similarily for the other equations.

Defining the dimensionless coefficients in the usual manner,

Froude Number

$$F_{r_0} = \frac{u_0^2}{g L_0} \qquad 5\text{-}98$$

Euler Number

$$E_{u_0} = \frac{L_0}{\rho_0 u_0^2}\left(\frac{\partial P}{\partial x}\right)_0 \qquad 5\text{-}99$$

and Reynolds Number

$$R_{E_0} = L_0 u_0 \rho_0 \div \eta_0 \qquad 5\text{-}100$$

Equation (5-97) becomes

$$\rho' a_x' = -\frac{\rho'}{F_{R_0}}\frac{\partial h'}{\partial x'} - E_{u_0}\left(\frac{\partial P}{\partial x}\right)' + \frac{1}{3R_{E_0}}\frac{\partial}{\partial x'}\left[\eta'\left(\frac{\partial u_x'}{\partial x'} + \frac{\partial u_y'}{\partial y'} + \frac{\partial u_z'}{\partial z'}\right)\right] + \frac{1}{R_{E_0}}\left[\frac{\partial}{\partial x'}\left(\eta'\frac{\partial u_x'}{\partial x'}\right) + \frac{\partial}{\partial y'}\left(\eta'\frac{\partial u_x'}{\partial y'}\right) + \frac{\partial}{\partial z'}\left(\eta'\frac{\partial u_x'}{\partial z'}\right)\right] \qquad 5\text{-}101$$

and similarily for the other equations.

In case the fluid is incompressible, the flow isothermal and the viscosity independent of any pressure changes which are encountered in the flow, Equation (5-101) becomes

$$a_x' = -\frac{1}{F_{r_0}}\frac{\partial h'}{\partial x'} - E_{u_0}\left(\frac{\partial P}{\partial x}\right)' + \frac{1}{R_{E_0}}\left[\frac{\partial^2 u_x'}{\partial x'^2} + \frac{\partial^2 u_x'}{\partial y'^2} + \frac{\partial^2 u_x'}{\partial z'^2}\right] \qquad 5\text{-}102$$

and similarily for the other equations. Equation (5-95) becomes

$$\frac{\partial u_x'}{\partial x'} + \frac{\partial u_y'}{\partial y'} + \frac{\partial u_z'}{\partial z'} = 0$$

Equations (5-101) and the other two which are similar to it for the Y and Z-components together with Equation (5-95) can be solved if sufficient restrictions are placed upon the flow situation so that one of the five unknown quantities (such as those mentioned on p 5-15)is eliminated as a variable, eg. if the temperature is assumed constant. A complete solution would give the values of all flow variables throughout the flow so that the values of the characteristic parameters could be determined. Hence the various dimensionless coefficients or ratios could be calculated and their interrelations determined throughout the flow from the equations. Thus the dimensionless equations which have been derived are equivalent to the dimensionless relations derived in Chapter I.

The preceeding analysis is more penetrating than that of Dimensional Analysis since except for the case summarized in Equations (5-102) and (5-103), there are variables (η and ρ) in addition to the unknowns, which are functions of state, and which vary in different manners for different fluids for equivalent changes in the state conditions. Thus no universal relation can be determined among the dimensionless ratios Re_o, Fr_o, and Eu_o which are true for all fluids.

5-13 Exact Solutions of the Equations of Motion for Idealized Laminar Flow[1]

Since only four of the five independent equations relating the five unknown quantities mentioned on p 5-15 have been derived, it is necessary to impose sufficient restrictions on the flow to eliminate one of the unknowns as a variable in order to be able to give a complete solution.

As an illustration of the application of the Equations of Motion derived thus far, the two cases of idealized laminar flow treated in Chapter I Sect. 11 and 12 will be reanalyzed.

1 See Sec. 1-8 for the definition of idealized flow.

Idealized Laminar Flow in a Circular Pipe

Fig. 5-18

From Equation (5-15a)

$$\frac{\partial\rho}{\partial\theta} + u_r\frac{\partial\rho}{\partial r} + \frac{u_\psi}{r}\frac{\partial\rho}{\partial\psi} + u_x\frac{\partial\rho}{\partial x} + \frac{\rho}{r}\left[\frac{\partial(r u_r)}{\partial r} + \frac{\partial u_\psi}{\partial\psi} + \frac{\partial(r u_x)}{\partial x}\right] = 0 = \frac{\partial\rho}{\partial\theta} + \frac{\rho}{r}\left[\frac{\partial(r u_r)}{\partial r} + \frac{\partial u_\psi}{\partial\psi} + \frac{\partial(r u_x)}{\partial x}\right] \qquad 5\text{-}104$$

Since ρ is a constant in idealized flow,

$$\frac{\partial(r u_r)}{\partial r} + \frac{\partial u_\psi}{\partial\psi} + \frac{\partial(r u_x)}{\partial x} = 0 \qquad 5\text{-}105$$

Since the flow is laminar, for this type of idealized flow, only u_x is not zero ie. there are no radial or azimuthal components of velocity. Therefore

$$\frac{\partial u_x}{\partial x} = 0 \qquad 5\text{-}106$$

Further, since $\Omega = 0$ for idealized flow also, Equation (5-76) becomes

$$0 = \frac{\partial P}{\partial r} \qquad 5\text{-}107$$

from Equation (5-105), and the fact that $u_r = 0$.

Similarily Equation (5-77) becomes

$$0 = \frac{\partial P}{\partial \varphi} \qquad 5\text{-}108$$

since $u_\varphi = 0$.

Since steady state conditions are assumed in idealized flow, from Equations (5-78) and (5-81)

$$\rho u_x \frac{\partial u_x}{\partial x} = -\frac{\partial P}{\partial x} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\eta r \frac{\partial u_x}{\partial r}\right) + r\frac{\partial}{\partial x}\left(\eta \frac{\partial u_x}{\partial x}\right)\right] \qquad 5\text{-}109$$

because by the symmetry of the flow conditions

$$\frac{\partial u_x}{\partial x} = 0 \qquad 5\text{-}110$$

However from Equation (5-106) and Equations (5-107) and (5-108), Equation (5-109) becomes

$$\frac{dP}{dx} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\eta r \frac{\partial u_x}{\partial r}\right)\right. \qquad 5\text{-}111$$

giving the relation between the pressure gradient and the viscous forces.

To solve Equation (5-111) integrate once with respect to r to obtain

$$\frac{r^2}{2}\frac{dP}{dx} = \eta r \frac{\partial u_x}{\partial r} + A(x) \qquad 5\text{-}112$$

where A(x) is an arbitrary function of x (corresponds to constant of integration in ordinary integration).

Since the flow is symmetrical about the axis,

$\frac{\partial u_x}{\partial r} = 0$ at $r = 0$, so that 5-113

$$A(x) = 0 \qquad 5\text{-}114$$

Since T is a constant and $\left(\frac{\partial \eta}{\partial P}\right)_T = 0$ by the assumptions of idealized flow[1], η is a constant also, and a second radial integration gives

$$\frac{r^2}{4}\frac{dP}{dx} = \eta u_x + B(x) \qquad 5\text{-}115$$

The velocity u_x is zero when $r = r_o$ so 5-116

$$B(x) = \frac{r_o^2}{4}\frac{dP}{dx}$$

5-117

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1 In case it is desired to make the restrictions as weak as possible, it is not necessary to assume $\partial\eta/\partial P)_T = 0$, since $\partial P/\partial r = 0$ and a radial integration would not be disturbed by a functional dependence of η on P. However Equation (5-112) would then involve a quantity η which was a function of total pressure and hence axial distance, x.

Thus finally

$$-\frac{r_0^2 - r^2}{4\eta} \cdot \frac{dP}{dx} = u_x \qquad 5\text{-}118$$

Comparison of Equation (5-118) with Equations (1-36) and (1-37) shows that the same results have been obtained.

In case the isothermal restriction is removed, while retaining all the other restrictions of idealized laminar flow, the results are identical as far as Equation (5-112). In case the viscosity is independent of temperature (a very artificial restriction) the above analysis is unmodified. In the more general case where η is a function of T, from Equation (5-112)

$$\frac{\partial u_x}{\partial r} = \frac{r}{2\eta} \frac{dP}{dx} + \frac{A(x)}{r} \qquad 5\text{-}119$$

so that in general Equation (5-114) must be true also.

This equation together with the last independent equation of motion to be derived, the Energy Equation (see Chapter 9) may be solved numerically.

<u>Idealized Laminar Flow Between Parallel Plates</u>

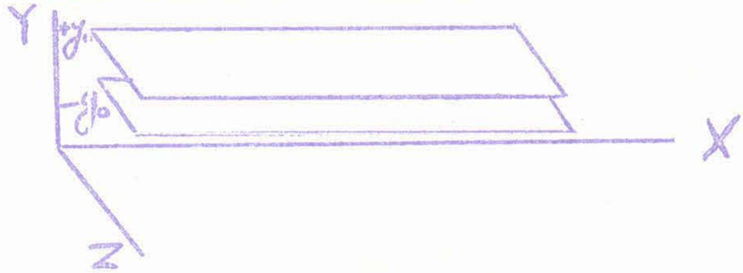

Fig. 5-19

From Equation (5-3)

$$\frac{dP}{d\theta} = -\rho\left[\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \qquad 5\text{-}120$$

Since ρ is a constant for idealized flow,

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0 \qquad 5\text{-}121$$

For laminar flow under these conditions $u_y = u_z = 0$ since the motion is wholly in the X-direction. Hence Equation (5-121) becomes

$$\frac{\partial u_x}{\partial x} = 0 \qquad 5\text{-}122$$

Further, since $\Omega = 0$ by assumption of idealized flow, Equation (5-71) becomes

$$\frac{\partial P}{\partial y} = 0 \qquad 5\text{-}123$$

and Equation (5-72)

$$\frac{\partial P}{\partial z} = 0 \qquad 5\text{-}124$$

Since steady state is assumed, from Equation (5-70) and (5-18)

$$\rho u_x \frac{\partial u_x}{\partial x} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(\eta \frac{\partial u_x}{\partial x}\right) + \frac{\partial}{\partial y}\left(\eta \frac{\partial u_x}{\partial y}\right) \qquad 5\text{-}125$$

because by symmetry

$$\frac{\partial u_x}{\partial z} = 0 \qquad 5\text{-}126$$

From Equation (5-122), and Equations (5-123) and (5-124), Equation (5-125) becomes

$$\frac{dP}{dx} = \frac{\partial}{\partial y}\left(\eta \frac{\partial u_x}{\partial y}\right) \qquad 5\text{-}127$$

Integrating once with respect to y,

$$y\frac{dP}{dx} = \eta \frac{\partial u_x}{\partial y} + A(x) \qquad 5\text{-}128$$

From symmetry of flow,

$$\frac{\partial u_x}{\partial y} = 0 \text{ when } y = 0 \qquad 5\text{-}129$$

so $A(x) = 0$ 5-130

Since T is a constant and $\left(\frac{\partial \eta}{\partial P}\right)_T = 0$ by assumptions of idealized flow, η is a constant[1], and a second integration with respect to y gives

$$\frac{y^2}{2}\cdot\frac{dP}{dx} = \eta u_x + B(x) \qquad 5\text{-}131$$

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1 See parallel footnote for circular pipe on p (5-57).

The velocity is zero at the walls ie.

$$u_x = 0 \quad \text{when } y = \pm y_0 \tag{5-132}$$

so that from Equation (5-131)

$$B(x) = y_0^2/2 \, \frac{dP}{dx} \tag{5-133}$$

and finally

$$-\frac{y_0^2 - y^2}{2\eta} \cdot \frac{dP}{dx} = u_x \tag{5-134}$$

Comparison of Equation (5-134) with Equations (1-46) and (1-47) shows that the same results have been obtained.

As in the previous case, if the isothermal restriction is removed, Equation (5-128) may be integrated numerically with the Energy Equation.

5-14 Initial and Boundary Conditions on the Equations of Motion

In the general case of the motion of a fluid, sufficient mathematical conditions must be given in order to obtain a definite solution to the Equations of Motion. Just what constitutes sufficient conditions is difficult to state from a purely mathematical standpoint, and they vary from situation to situation, yet it generally possible to ascertain the conditions from physical reasoning. Thus in the last section the physical situation at the boundaries to the flow gave the mathematical condition that the relative velocity at the wall was zero. In the general case, the physical situation at the boundaries to the flow is generally known and may be made to yield the mathematical "boundary" conditions, and the physical situation throughout the flow is generally known at some instant (usually $\theta = 0$) giving the mathematical "initial" conditions. (The latter conditions did not appear in the analysis of the last section because the motion was steady.) If the physical analysis is adequate, these physical conditions will determine the complete behavior of the fluid, and hence furnish sufficient mathematical conditions on the equations of motion to yield a definite solution. For illustrations of the determination and use of boundary and initial conditions, see Chapters 6 and 10 and Lamb, Goldstein, and Boelter[1].

5-15 General Discussion of Laminar Flow

Very few other exact solutions of the Equations of Motion have been derived[2], although a number of approximate solutions have been obtained by neglecting various terms in the equations[3]. The chief mathematical difficulty which has prevented the obtaining of an exact solution in the general case lies in the fact that in the Equation of Continuity (5-3) and the Stokes-Navier Equations (5-18), (5-19), (5-20), (5-70), (5-71), (5-72) terms appear involving the product of two unknown quantities ($u_x \frac{\partial u_z}{\partial x}$, for example), or as is said in mathematics, the equations are non-linear; (the term linear equation being restricted to those in which every term contains at most but one unknown quantity to the first power only.) This fact prevents the use of either of two of the very few general techniques of solving partial differential equations exactly (or

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1 L.M.K. Boelter, V.H. Cherry, H. A. Johnson, Heat Transfer, Univ. of California. Hereafter Boelter.

2 See Lamb and Goldstein.

3 See Lamb and Goldstein.

as is said in mathematics, analytically): the method of separation of variables[1], and the method of the Laplace Transform[2]. Many of the approximate solutions which have been obtained were secured by neglecting all the non-linear terms in the Equations of Motion and any others that were necessary, so that these techniques could be applied.

Since it occasionally occurs that the Equations of Motion may be legitimately simplified to permit the use of these techniques, their range of applicability will be outlined so that the texts given in the footnotes may be consulted for details.

The method of separation of variables may be applied to homogeneous linear partial differential equations. (A homogeneous linear equation is one that contains no terms not involving an unknown quantity to the first power only.) In addition the parameters of the equation must be expressible as products of factors each of which is a function of only one independent variable. (See Chapter VI for an example).

The method of the Laplace Transform may be applied to linear partial differential equations with only rather weak restrictions on the parameters.

There are many special methods for solving particular partial differential equations, and many partial differential equations may be greatly simplified by suitable transformations or "mathematical tricks". Excellent discussions of methods of attack on partial differential equations are given in Frank and von Mises, Bateman[3], and Courant and Hilbert[4].

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1 R. V. Churchill, Fourier Series and Boundary Value Problems, McGraw Hill Company, (1941).

2 R. V. Churchill, R. V. Churchill Modern Operational Mathematics in Engineering, McHraw Hill (1944).
H. S. Carslaw and J. C. Jaeger, Operational Methods in Applied Mathematics, Oxford (1941).

3 H. Bateman, Partial Differential Equations of Mathematical Physics, Dover (1944).

4 R. Courant and D. Hilbert, Die Methoden der Mathematischen Physik. Edwards Bros.

In general the exact or analytic solution to linear partial differential equations involves either infinite series or complicated integrals, both involving coefficients which are often extremely difficult to evaluate numerically. Consequently only approximate numerical answers can be obtained, as a rule. Further the partial differential equations which were to be solved must often be unjustifiably simplified to yield linear equations with constant coefficients or coefficients which are artificially prescribed so that the equations may be solved. Finally, if any boundary surfaces other than plane, circular cylindrical, spherical, or a few others are introduced, the exact solution becomes much more difficult if not impossible.

On the other hand, there is an approximate numerical method of solving partial differential equations which is being rapidly developed at the present time. This method makes no restriction on the form of the partial differential equation to be solved, can allow for any necessary variation in the parameters (such as in η with T, for example) and can yield answers of any desired degree of accuracy depending on the time expended on the solution. In common with all numerical methods of solution, however, only a single physical problem may be solved at a time, and general trends, optimum conditions, etc. can only be obtained by solving many separate problems individually. This method is outlined in Chapter 6 and consists essentially in replacing the given partial differential equation with a partial finite difference equation. If an approximate analytic solution to the problem can be obtained without too much difficulty, it is often more efficient to consider that the "exact" solution is the sum of the analytic approximation and a further approximation obtained by numerical means as suggested above i.e. that the numerical process be applied to "residual" quantities only.

In view of the breakdown[1] of laminar flow to turbulent flow for Reynold's numbers below those usually encountered in chemical engineering technology (i.e. at about 2000 for idealized flow in a circular pipe, and for the case of a compressible fluid being heated or cooled in a horizontal circular pipe, even for a fluid which is stationary on the average, because of convection currents) even this general method of solution is not of too great practical value since it is virtually impossible to solve the Stokes-Navier Equations exactly for turbulent flow. The difficulty arises from the fact that turbulent flow continuously generates eddies which are gradually dissipated as they move through the fluid. An exact solution for the case of steady flow, for example, would require the calculation of the history of all the eddies formed from an initial disturbance (eg. that disturbance which caused the transition from laminar to turbulent flow) as well as of all the eddies formed by the eddies, etc., until the average conditions become constant throughout the flow.

However, for sufficiently viscous fluids, narrow conduits, and low velocities, i.e. for small enough Reynold's numbers, laminar flow is usually stable, so that many practical problems do arise in chemical engineering technology which involve laminar flow.

5-16 <u>General Discussion of Turbulent Flow</u>

There are two broad types of attack on the problem of turbulence being advanced today.

The first, the statistical approach, is the more difficult and less developed of the two, though ultimately it promises to yield an accurate solution to the problem. It ignores the detailed history of the individual eddy motions

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1 See Sect. 1-14.

in turbulent flow, which, as previously discussed, is hopelessly complex. Instead, the statistical approach assumes that turbulent flow may be described by distributions of velocities, temperatures, and pressures at any point in the flow at any instant; the distributions will vary from point to point and instant to instant, in general. By a distribution of velocities, for example, is meant a functional relationship between a velocity and the probability of its occurrence or between a velocity fluctuation and the probability of its occurrence, where the probability depends on position, and in the case of unsteady flow, on time as well. From the probabilities, the average conditions can be calculated. For any given probability, the velocities, pressures, and temperatures given by the distributions must satisfy the Equations of Motion, and the distributions themselves must satisfy the laws of probability.

An important variation of the statistical theory of turbulence considers, not the distributions of velocities at a particular point, but the correlation between the velocities at two or more points at any instant. For a further discussion of this method of approach see Chapter II and "A Review of the Statistical Theory of Turbulence" by H. L. Dryden Quart App. Math. I, 7-42, (1943) which also contains a bibliography of other statistical theories.

The second approach to the problem of turbulent flow consists in making some assumptions as to the mechanism of turbulence. The Prandtl mixing length hypothesis, for example, assumes that turbulence consists essentially in the sudden motion of masses of fluid from one layer in the fluid to another equidistant from it, that the masses of fluid suffer no changes in transit, and that they give up their properties that they had in the first layer on striking the second, and assume the properties of the second layer. See Chapters II and III and Goldstein for further discussion.

5-17 Reynolds[1] Transformation of the Equations of Motion

Reynolds introduced a transformation of the Equations of Motion which is of great importance in turbulence theories because it separates the effects caused by average conditions from those produced by fluctuations. This transformation utilizes the concepts of average and fluctuation quantities introduced in Chapter I which are redefined here for the average, instantaneous, and fluctuation values of a variable G

$$G = \bar{G} + G_f \qquad 5\text{-}135$$

$$\bar{G} = \frac{1}{\Theta}\int_{\theta_0 - \frac{1}{2}\Theta}^{\theta_0 + \frac{1}{2}\Theta} G\,d\theta \qquad 5\text{-}136$$

where G is the instantaneous value of the variable at a given point at time θ_0, $\bar{G}$ is the (time) average value, and G_f is the fluctuation value. The period of the integration, Θ is taken long enough that $\bar{G}$ is not sensibly changed by taking a longer time. In steady flow, Θ, may be taken as long as desired, but as discussed in Chapter I, the concept of a fluctuation value can be applied to unsteady flow only if the general conditions are changing very slowly compared to the fluctuations. (In the case of unsteady flow, the definition of a fluctuation value can be improved by repeating the experiment so that a shorter period of integration can be used, employing the augmented data.)

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1 O. Reynolds, Phil. Trans. A186, 123 (1894).

Before performing the Reynolds' transformation it is necessary to obtain some useful relations concerning average quantities:

From Equation (5-135) and (5-136), it follows that the average of a fluctuation value is zero

$$\overline{G_f} = 0 \qquad 5\text{-}137$$

and the double average is the average,

$$\overline{\overline{G}} = \overline{G} \qquad 5\text{-}138$$

where G is any variable. It I is any variable (including possibly G), from Equation (5-137)

$$\overline{\overline{G}\, I_f} = 0 \qquad 5\text{-}139$$

since $\overline{G}$ is constant; and hence

$$\overline{GI} = \overline{(\overline{G}+G_f)(\overline{I}+I_f)} = \overline{G}\,\overline{I} + \overline{G_f I_f} \qquad 5\text{-}140$$

It is possible to invert the order of a partial differentiation with respect to one variable and a partial integration with respect to another under broad continuity conditions.[1] Consequently

$$\overline{\frac{\partial G}{\partial L}} = \frac{1}{\Theta}\int_{\theta_0-\frac{1}{2}\theta}^{\theta_0+\frac{1}{2}\theta} \frac{\partial G}{\partial L}\,d\theta = \frac{\partial}{\partial L}\cdot\frac{1}{\Theta}\int_{\theta_0-\frac{1}{2}\theta}^{\theta_0+\frac{1}{2}\theta} G\,d\theta = \frac{\partial \overline{G}}{\partial L} \qquad 5\text{-}141$$

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1 See Wilson, p-283, Burrington p-298.

or the average of the partial of G with respect to a distance is the partial of the average of G with respect to the distance.

If the partial derivative is taken with respect to time, the same result is obtained for slightly different reasons. Thus

$$\overline{\frac{\partial G}{\partial \theta}} = \frac{1}{\theta}\int_{\theta_0-\frac{1}{2}\theta}^{\theta_0+\frac{1}{2}\theta}\frac{\partial G}{\partial \theta}d\theta = \frac{1}{\theta}\left[G\left(\theta_0+\frac{\theta}{2}\right)-G\left(\theta_0-\frac{\theta}{2}\right)\right.$$

$$= \frac{1}{\theta}\frac{\partial}{\partial \theta}\int_{\theta_0-\frac{1}{2}\theta}^{\theta_0+\frac{1}{2}\theta} G\,d\theta = \frac{\partial}{\partial \theta}\frac{1}{\theta}\int_{\theta_0-\frac{1}{2}\theta}^{\theta_0+\frac{1}{2}\theta} G\,d\theta = \frac{\partial \overline{G}}{\partial \theta} \qquad 5\text{-}142$$

because the second member is obtained from the third by the theorem for differentiating integrals[1] and $\frac{1}{\theta}$ & $\frac{\partial}{\partial \theta}$ may be interchanged in the fourth member since $\frac{1}{\theta}$ is a constant as far as this differentiation is concerned.

The Reynolds' transformation consists in substituting the sum of the average and fluctuation values of all dependent variables for the instantaneous values in the Equations of Motion and then taking the time average of the Equations term by term, by Equation (5-136). This procedure yields the type of information generally desired about turbulent flow since only the average conditions are usually of interest, and the transformation permits the detailed flow processes to be ignored provided certain fluctuation quantities can be calculated.

Thus if the average and fluctuation values of the density and the velocity components of the fluid are substituted for the instantaneous values in the Equation of Continuity, Equation (5-3), i.e. if

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1. See the same calculus texts pp 27 and p 298 respectively.

$$\rho = \bar{\rho} + \rho_f$$
$$u_x = \bar{u}_x + u_{xf}$$
$$u_y = \bar{u}_y + u_{yf} \qquad 5\text{-}14$$
$$u_z = \bar{u}_z + u_{zf}$$

are substituted in Equation (5-3), there results

$$\frac{d(\bar{\rho}+\rho_f)}{d\theta} = -(\bar{\rho}+\rho_f)\left[\frac{\partial \bar{u}_x}{\partial x} + \frac{\partial \bar{u}_y}{\partial y} + \frac{\partial \bar{u}_z}{\partial z} + \frac{\partial u_{xf}}{\partial x} + \frac{\partial u_{yf}}{\partial y} + \frac{\partial u_{zf}}{\partial z}\right] = \frac{\partial(\bar{\rho}+\rho_f)}{\partial \theta} + (\bar{u}_x + u_{xf})\frac{\partial(\bar{\rho}+\rho_f)}{\partial x}$$
$$+ (\bar{u}_y + u_{yf})\frac{\partial(\bar{\rho}+\rho_f)}{\partial y} + (\bar{u}_z + u_{zf})\frac{\partial(\bar{\rho}+\rho_f)}{\partial z} \qquad 5\text{-}144$$

The average of Equation (5-144) on applying Equation (5-136) term by term and simplifying with Equations (5-142), (141), (140), (139), (138), and (137) is

$$\overline{\frac{d(\bar{\rho}+\rho_f)}{d\theta}} = -\bar{\rho}\left[\frac{\partial \bar{u}_x}{\partial x} + \frac{\partial \bar{u}_y}{\partial y} + \frac{\partial \bar{u}_z}{\partial z}\right] - \overline{\rho_f\left[\frac{\partial u_{xf}}{\partial x} + \frac{\partial u_{yf}}{\partial y} + \frac{\partial u_{zf}}{\partial z}\right]} = \frac{\partial \bar{\rho}}{\partial \theta} + \bar{u}_x \frac{\partial \bar{\rho}}{\partial x} + \bar{u}_y \frac{\partial \bar{\rho}}{\partial y} + \bar{u}_z \frac{\partial \bar{\rho}}{\partial z}$$
$$+ \overline{u_{xf}\frac{\partial \rho_f}{\partial x}} + \overline{u_{yf}\frac{\partial \rho_f}{\partial y}} + \overline{u_{zf}\frac{\partial \rho_f}{\partial z}} \qquad 5\text{-}145$$

Note that

$$\overline{\frac{dG}{d\theta}} \neq \frac{d\overline{G}}{d\theta} \qquad 5-146$$

The correct relations are shown between the first and third members of Equation (5-145).

If the fluid is incompressible, since $\frac{d\rho}{d\theta} = 0$ & $\rho' = 0$, Equation (5-145) reduces to

$$\frac{\partial \bar{u}_x}{\partial x} + \frac{\partial \bar{u}_y}{\partial y} + \frac{\partial \bar{u}_z}{\partial z} = 0 \qquad 5-147$$

which is the average of the Equation of Continuity for an incompressibble fluid.

Before performing the Reynolds' transformation on the Stokes-Navier Equations, it is desirable to transform them by means of a relation which will now be derived, in order to obtain quantities useful in mixing length turbulence theories.

If ξ is any arbitrary finite and continuous function of x, y, z, and θ with finite and continuous derivatives, by the general theorem for partial differentiation.

$$\frac{d(\rho\xi)}{d\theta} = \frac{\partial(\rho\xi)}{\partial\theta} + u_x \frac{\partial(\rho\xi)}{\partial x} + u_y \frac{\partial(\rho\xi)}{\partial y} + u_z \frac{\partial(\rho\xi)}{\partial z}$$

$$= \rho \frac{d\xi}{d\theta} + \xi \frac{d\rho}{d\theta} \qquad 5-148$$

From the Equation of Continuity, Equation (5-3), Equation (5-148) may be transformed into

$$\rho \frac{d\xi}{d\theta} = \xi\rho\left[\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right] + \frac{\partial(\rho\xi)}{\partial\theta} + u_x\frac{\partial(\rho\xi)}{\partial x} + u_y\frac{\partial(\rho\xi)}{\partial y} + u_z\frac{\partial(\rho\xi)}{\partial z} \qquad 5\text{-}149$$

or

$$\rho\frac{d\xi}{d\theta} = \frac{\partial(\rho\xi)}{\partial\theta} + \frac{\partial(\rho u_x\xi)}{\partial x} + \frac{\partial(\rho u_y\xi)}{\partial y} + \frac{\partial(\rho u_z\xi)}{\partial z} \qquad 5\text{-}150$$

by the rule for differentiating a product.

If ξ is taken as u_x, u_y, and u_z in turn and the results substituted in Equations (5-28), (5-29), and (5-30) respectively, the Momentum Equations, the results are

$$\frac{\partial(\rho u_x)}{\partial\theta} = \rho\Phi_x + \frac{\partial}{\partial x}(T_{xx} - \rho u_x u_x) + \frac{\partial}{\partial y}(T_{yx} - \rho u_y u_x) + \frac{\partial}{\partial z}(T_{zx} - \rho u_z u_x) \qquad 5\text{-}151$$

$$\frac{\partial(\rho u_y)}{\partial\theta} = \rho\Phi_y + \frac{\partial}{\partial x}(T_{xy} - \rho u_x u_y) + \frac{\partial}{\partial y}(T_{yy} - \rho u_y u_y) + \frac{\partial}{\partial z}(T_{zy} - \rho u_z u_y) \qquad 5\text{-}152$$

only relatively slowly so that larger fluctuations in pressure and temperature may be expected.

For an incompressible fluid or a fluid which is incompressible as far as the turbulent fluctuations are concerned, the Momentum Equations become

$$\frac{\partial(\bar{\rho}\bar{u}_x)}{\partial\theta} = \bar{\rho}\bar{\Phi}_x + \frac{\partial}{\partial x}\left(\bar{\tau}_{xx} - \bar{\rho}\,\bar{u}_x\bar{u}_x - \bar{\rho}\,\overline{u_{xf}\,u_{xf}}\right) + \frac{\partial}{\partial y}\left(\bar{\tau}_{yx} - \bar{\rho}\,\bar{u}_y\bar{u}_x - \bar{\rho}\,\overline{u_{yf}\,u_{xf}}\right) + \frac{\partial}{\partial z}\left(\bar{\tau}_{zx} - \bar{\rho}\,\bar{u}_z\bar{u}_x - \bar{\rho}\,\overline{u_{zf}\,u_{xf}}\right) \qquad 5\text{-}154$$

$$\frac{\partial(\bar{\rho}\bar{u}_y)}{\partial\theta} = \bar{\rho}\,\bar{\Phi}_y + \frac{\partial}{\partial x}\left(\bar{\tau}_{xy} - \bar{\rho}\,\bar{u}_x\bar{u}_y - \bar{\rho}\,\overline{u_{xf}u_{yf}}\right) + \frac{\partial}{\partial y}\left(\bar{\tau}_{yy} - \bar{\rho}\,\bar{u}_y\bar{u}_y - \bar{\rho}\,\overline{u_{yf}\,u_{yf}}\right) + \frac{\partial}{\partial z}\left(\bar{\tau}_{zy} - \bar{\rho}\,\bar{u}_z\bar{u}_y - \bar{\rho}\,\overline{u_{zf}u_{yf}}\right) \qquad 5\text{-}155$$

$$\frac{\partial(\bar{\rho}\bar{u}_z)}{\partial\theta} = \bar{\rho}\,\bar{\Phi}_z + \frac{\partial}{\partial x}\left(\bar{\tau}_{xz} - \bar{\rho}\,\bar{u}_x\bar{u}_z - \bar{\rho}\,\overline{u_{xf}u_{zf}}\right) + \frac{\partial}{\partial y}\left(\bar{\tau}_{yz} - \bar{\rho}\,\bar{u}_y\bar{u}_z - \bar{\rho}\,\overline{u_{yf}u_{zf}}\right) + \frac{\partial}{\partial z}\left(\bar{\tau}_{zz} - \bar{\rho}\,\bar{u}_z\bar{u}_z - \bar{\rho}\,\overline{u_{zf}u_{zf}}\right) \qquad 5\text{-}156$$

(It is assumed that the external field is constant with respect to time.) For incompressible fluids, the bars may be omitted from the densities.

For a fluid which is essentially incompressible as far as turbulent fluctuations are concerned, the Equation of Continuity becomes

$$-\bar{\rho}\left[\frac{\partial\bar{u}_x}{\partial x} + \frac{\partial\bar{u}_y}{\partial y} + \frac{\partial\bar{u}_z}{\partial z}\right] = \frac{\partial\bar{\rho}}{\partial\theta} + \bar{u}_x\frac{\partial\bar{\rho}}{\partial x} + u_y\frac{\partial\bar{\rho}}{\partial y} + u_z\frac{\partial\bar{\rho}}{\partial z}. \qquad 5\text{-}157$$

The equations (5-154), (5-155), and (5-156) have the same form as the instantaneous Stokes-Navier Equations (Equation (5-151), (5-152), and (5-153)) when the instantaneous values of the variables are replaced by average values, if the viscous stresses τ_{ij} are changed to $\bar{\tau}_{ij} - \bar{\rho} \cdot \overline{u_{if} u_{jf}}$ The latter term of this stress is known as the Reynolds' stress and arises from the transport of momentum by the velocity fluctuations just as τ_{ij} arises from the transport of momentum by molecular agitation[1].

Cylindrical Coordinates

The Reynolds' transformation of the Equations of Motion in cylindrical coordinates yields, for the Equation of Continuity of an incompressible fluid

$$\frac{1}{r}\left[\frac{\partial(r\bar{u}_r)}{\partial r} + \frac{\partial \bar{u}_\psi}{\partial \psi} + \frac{\partial(r\bar{u}_x)}{\partial x}\right] = 0 \qquad 5\text{-}158$$

For a fluid which is only incompressible as far as the turbulent fluctuations are concerned, the Equation of Continuity becomes

$$\frac{\partial \bar{\rho}}{\partial \theta} + \frac{1}{r}\left[\frac{\partial(r\bar{\rho}\bar{u}_r)}{\partial r} + \frac{\partial(\bar{\rho}\bar{u}_\psi)}{\partial \psi} + \frac{\partial(r\bar{\rho}\bar{u}_x)}{\partial x}\right] = 0 \qquad 5\text{-}159$$

The Momentum Equations in cylindrical coordinates take the forms

$$\rho\frac{du_r}{d\theta} = \rho\Phi_r + \frac{1}{r}\left(\frac{\partial(r\tau_{rr})}{\partial r} + \frac{\partial \tau_{\psi r}}{\partial \psi} + \frac{\partial(r\tau_{xr})}{\partial x}\right) \qquad 5\text{-}160$$

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1 See Millikan, Roller and Watson, Mechanics, Molecular Physics, Heat and Sound, Ginn and Co. (1937) p-216.

$$\rho \frac{du_\psi}{d\theta} = \rho \Phi_\psi + \frac{1}{r}\left(\frac{\partial (r T_{r\psi})}{\partial r} + \frac{\partial T_{\psi\psi}}{\partial \psi} + \frac{\partial (r T_{x\psi})}{\partial x}\right) \quad 5\text{-}161$$

and

$$\rho \frac{du_x}{d\theta} = \rho \Phi_x + \frac{1}{r}\left(\frac{\partial (r T_{rx})}{\partial r} + \frac{\partial T_{\psi x}}{\partial \psi} + \frac{\partial (r T_{xx})}{\partial x}\right) \quad 5\text{-}162$$

While the cylindrical analogue of Equation (5-150) is

$$\rho \frac{d\xi}{d\theta} = \frac{\partial (\rho \xi)}{\partial \theta} + \frac{1}{r}\left(\frac{\partial (r \rho u_r \xi)}{\partial r} + \frac{\partial (\rho u_\psi \xi)}{\partial \psi} + \frac{\partial (r \rho u_x \xi)}{\partial x}\right) \quad 5\text{-}163$$

Taking ξ as u_r, u_ψ and u_x in turn and substituting the results in Equations (5-160), (5-161), and (5-162) respectively, there results

$$\frac{\partial (\rho u_r)}{\partial \theta} = \rho \Phi_r + \frac{1}{r}\left[\frac{\partial}{\partial r} r (T_{rr} - \rho u_r u_r) + \frac{\partial}{\partial \psi}(T_{\psi r} - \rho u_\psi u_r) + \frac{\partial}{\partial x} r (T_{xr} - \rho u_x u_r)\right] \quad 5\text{-}164$$

$$\frac{\partial (\rho u_\psi)}{\partial \theta} = \rho \Phi_\psi + \frac{1}{r}\left[\frac{\partial}{\partial r} r (T_{r\psi} - \rho u_r u_\psi) + \frac{\partial}{\partial \psi}(T_{\psi\psi} - \rho u_\psi u_\psi) + \frac{\partial}{\partial x} r (T_{x\psi} - \rho u_x u_\psi)\right] \quad 5\text{-}165$$

$$\frac{\partial(\rho u_x)}{\partial\theta} = \rho\Phi_x + \frac{1}{r}\left(\frac{\partial}{\partial r} r(T_{rx} - \rho u_r u_x) + \frac{\partial}{\partial\psi}(T_{\psi x} - \rho u_\psi u_x) + \frac{\partial}{\partial x}(T_{xx} - \rho u_x u_x)\right) \qquad 5\text{-}166$$

If the Reynolds' transformation is performed on these Momentum Equations, the result is, for a fluid which is incompressible as far as turbulent fluctuations are concerned:

$$\frac{\partial(\bar\rho\bar u_r)}{\partial\theta} = \bar\rho\Phi_r + \frac{1}{r}\left(\frac{\partial}{\partial r} r(\bar T_{rr} - \bar\rho \bar u_r \bar u_r - \bar\rho\,\overline{u_{rf} u_{rf}}) + \frac{\partial}{\partial\psi}(\bar T_{\psi r} - \bar\rho \bar u_\psi \bar u_r - \bar\rho\,\overline{u_{\psi f} u_{rf}}) + \frac{\partial}{\partial x} r(\bar T_{xr} - \bar\rho \bar u_x \bar u_r - \bar\rho\,\overline{u_{xf} u_{rf}})\right) \qquad 5\text{-}167$$

$$\frac{\partial(\bar\rho\bar u_\psi)}{\partial\theta} = \bar\rho\Phi_\psi + \frac{1}{r}\left(\frac{\partial}{\partial r} r(\bar T_{r\psi} - \bar\rho \bar u_r \bar u_\psi - \bar\rho\,\overline{u_{rf} u_{\psi f}}) + \frac{\partial}{\partial\psi}(\bar T_{\psi\psi} - \bar\rho \bar u_\psi u_\psi - \bar\rho\,\overline{u_{\psi f} u_{\psi f}}) + \frac{\partial}{\partial x} r(\bar T_{x\psi} - \bar\rho \bar u_x \bar u_\psi - \bar\rho\,\overline{u_{xf} u_{\psi f}})\right) \qquad 5\text{-}168$$

$$\frac{\partial(\bar\rho\bar u_x)}{\partial\theta} = \bar\rho\Phi_x + \frac{1}{r}\left(\frac{\partial}{\partial r} r(\bar T_{rx} - \bar\rho \bar u_r \bar u_x - \bar\rho\,\overline{u_{rf} u_{xf}}) + \frac{\partial}{\partial\psi}(\bar T_{\psi x} - \bar\rho \bar u_\psi \bar u_x - \bar\rho\,\overline{u_{\psi f} u_{xf}}) + \frac{\partial}{\partial x} r(\bar T_{xx} - \bar\rho \bar u_x \bar u_x - \bar\rho\,\overline{u_{xf} u_{xf}})\right) \qquad 5\text{-}169$$

For an incompressible fluid, the density is constant and the bar may be removed.

<u>Application of the Reynolds' Transformation of the Equations of Motion to the Case of Idealized Flow between Parallel Plates</u>

V

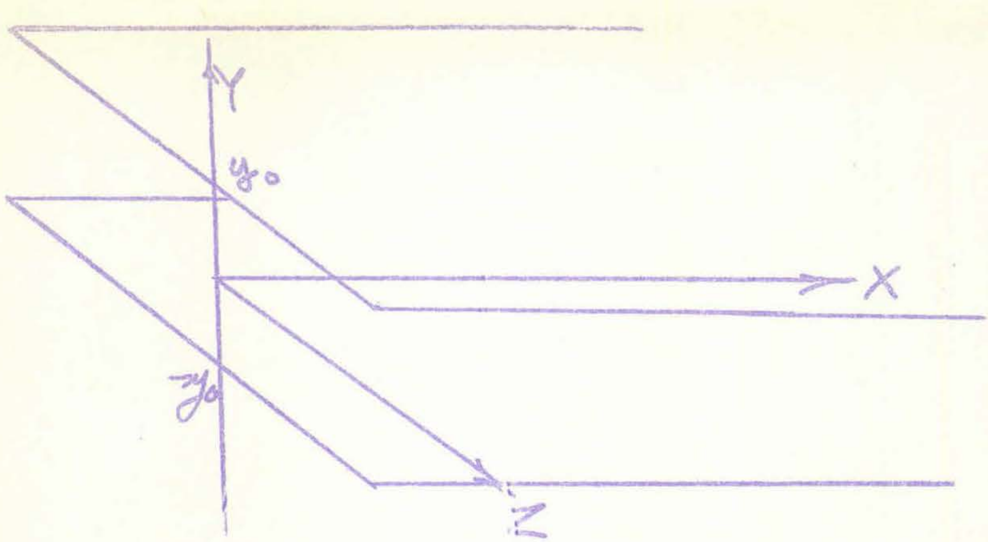

Fig. 5-20

For idealized turbulent flow between parallel plates spaced $2y_0$ apart (See Figure 5-20), the average velocities in the Y and Z-directions are zero since the axes are oriented so that the mean motion is in the X-direction i.e.

$$\bar{u}_y = \bar{u}_z = 0 \qquad 5\text{-}170$$

Since the flow is steady and the fluid incompressible

$$\frac{\partial(\bar{u}_x \rho)}{\partial \theta} = \frac{\partial(\bar{u}_y \rho)}{\partial \theta} = \frac{\partial(u_z \rho)}{\partial \theta} \qquad 5\text{-}171$$

Since the external fields were neglected

$$\overline{\Phi}_x = \overline{\Phi}_y = \overline{\Phi}_z = 0 \qquad 5\text{-}172$$

The average conditions in the flow are presumably independent of x and z so

$$\frac{\partial \rho \bar{u}_x \bar{u}_x}{\partial x} = \frac{\partial (\rho \overline{u_{xf} u_{xf}})}{\partial x} = \frac{\partial (\rho \overline{u_{zf} u_{xf}})}{\partial z} = \frac{\partial (\rho \overline{u_{xf} u_{yf}})}{\partial x}$$

$$= \frac{\partial}{\partial z}(\rho \overline{u_{zf} u_{yf}}) = \frac{\partial}{\partial x}(\rho \overline{u_{xf} u_{zf}}) = \frac{\partial}{\partial z}(\rho \overline{u_{zf} u_{zf}}) = 0 \qquad 5\text{-}173$$

Even if the independence of the average conditions from x & z is not employed, the third, fifth and sixth members are zero since

$$\overline{u_{zf} u_{xf}} = \overline{u_{zf} u_{yf}} = 0 \qquad 5\text{-}174$$

because the z fluctuations in velocity are independent of the x and y fluctuations; that is, for a given x or y fluctuation, the probability of a given positive or negative z fluctuation is equal so that the average is zero.

From Equations (5-57), (5-58), and (5-59), the Reynolds transformation yields

$$\overline{\tau}_{zx} = \overline{\tau}_{xz} = 0 \qquad 5\text{-}175$$

$$\overline{\tau_{zy}} = \overline{\tau_{yz}} = 0 \qquad 5\text{-}176$$

$$\overline{\tau_{xy}} = \overline{\tau_{yx}} = \bar{\eta}\frac{\partial \bar{u}_x}{\partial y} \qquad 5\text{-}177$$

provided the pressure and especially the temperature fluctuationsare not severe enough to cause appreciable changes in the viscosity.

Therefore Equation (5-154) becomes

$$\frac{\partial}{\partial x}\overline{\tau_{xx}} + \frac{\partial}{\partial y}\left(\overline{\tau_{yx}} - \rho\,\overline{u_{yf}u_{xf}}\right) = 0 \qquad 5\text{-}178$$

Equation (5-155) becomes

$$\frac{\partial}{\partial y}\left(\overline{\tau_{yy}} - \rho\,\overline{u_{yf}u_{yf}}\right) = 0 \qquad 5\text{-}179$$

since

$$\frac{\partial}{\partial x}\left(\overline{\tau_{xy}}\right) = \frac{\partial}{\partial x}\left(\bar{\eta}\frac{\partial \bar{u}_x}{\partial y}\right) = 0 \qquad 5\text{-}180$$

because of the independence of average conditions from x.

Similarily Equation (5-156) becomes

$$\frac{\partial}{\partial z}\overline{\tau_{zz}} = 0 \qquad 5\text{-}181$$

From Equations (5-63), (5-65), and (5-66), since the fluid is incompressible

$$\overline{T}_{xx} = -\overline{P} + 2\overline{\eta}\frac{\partial \overline{u}_x}{\partial x} \qquad 5\text{-}182$$

$$\overline{T}_{yy} = -\overline{P} \qquad 5\text{-}183$$

$$\overline{T}_{zz} = -\overline{P} \qquad 5\text{-}184$$

Either from the independence of conditions from x, or the Equation of Continuity, Equation (5-147)

$$\frac{\partial \overline{u}_x}{\partial x} = 0 \qquad 5\text{-}185$$

Consequently

$$\overline{T}_{xx} = \overline{T}_{yy} = \overline{T}_{zz} = -\overline{P} \qquad 5\text{-}186$$

Since Equation (5-186) shows that

$$\frac{\partial \overline{P}}{\partial y} + \frac{\partial}{\partial y}(\rho\overline{u_{yf}u_{yf}}) = 0 \qquad 5\text{-}187$$

from Equation (5-179), on differentiating Equation (5-187) with respect to x and reversing the order of differentiation

$$\frac{\partial}{\partial y}\left(\frac{\partial \overline{P}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial(\rho\overline{u_{yf}u_{yf}})}{\partial x}\right) = 0 \qquad 5\text{-}188$$

However, from the independence condition,

$$\frac{\partial\left(\rho\,\overline{u_{yf}\,u_{yf}}\right)}{\partial x}=0 \qquad 5-189$$

So

$$\frac{\partial}{\partial y}\left(\frac{\partial P}{\partial x}\right)=0 \qquad 5-190$$

On differentiating Equation (5-181) with respect to x, reversing the order, and substituting Equation (5-186), the result is

$$\frac{\partial}{\partial z}\left(\frac{\partial \bar{P}}{\partial x}\right)=0$$

so that $\frac{\partial \bar{P}}{\partial x}$ is a function only of x, if that. Also

$$\frac{\partial \bar{P}}{\partial z}=0 \qquad 5-192$$

Equation (5-178) becomes, on substituting Equations (5-186) and (5-177)

$$\frac{\partial \bar{P}}{\partial x}+\frac{\partial}{\partial y}\left(\bar{\eta}\frac{\partial \bar{u}_x}{\partial y}\right)-\rho\,\overline{u_{yf}\,u_{xf}}\Big)=0 \qquad 5-193$$

Integrating with respect to y, since $\frac{\partial \bar{P}}{\partial x}$ is not a function of y,

$$\rho\,\overline{u_{yf}\,u_{xf}}=y\frac{\partial \bar{P}}{\partial x}+\bar{\eta}\frac{\partial \bar{u}_x}{\partial y}+A(x) \qquad 5-194$$

Where $A(x)$ is an arbitrary function of x. When $y = 0$, for a given fluctuation in u_x there is an equal probability of a positive or negative fluctuation in u_y because of the symmetry of the flow, so that the average of $u_{yf}u_{xf}$ there is zero.

Hence

$$A(x) \equiv 0 \qquad 5\text{-}195$$

since

$$\frac{\partial \bar{u}_x}{\partial y} = 0 \quad at \; y = 0 \qquad 5\text{-}196$$

by symmetry

Therefore

$$\rho \, \overline{u_{yf} u_{xf}} = y \frac{\partial \bar{P}}{\partial x} + \bar{\eta} \frac{\partial \bar{u}_x}{\partial y} \qquad 5\text{-}197$$

Compare this relation with Equations (1-20) and (2-7).

Differentiation of Equation (5-193) with respect to x shows that

$$\frac{\partial^2 \bar{P}}{\partial x^2} = 0 \qquad 5\text{-}198$$

so that $\frac{\partial \bar{P}}{\partial x}$ is a constant.

CHAPTER VI

MOLECULAR THERMAL TRANSFER OF ENERGY

OR MOLECULAR CONDUCTION OF HEAT

Introduction

The transfer of energy in a material as a result of temperature gradients will be discussed in this chapter. It will be assumed that no macroscopic movement of the material occurs during the course of the process and the effect of radiation is negligible. It will be further assumed that transfers of energy by other than thermal means do not occur.

It is possible to consider transfers of energy which result from temperature gradients from either the macroscopic or the microscopic point of view. The former, which was first considered by Fourier[1], leads to predictions of the temperature of the material as a function of position and time, if sufficient conditions are imposed on the system to determine its behavior. The latter, which was first considered by Drude[2], has been mostly used to determine the relation between the thermal conductivity k of equation (6-1) and other physical quantities.

- -

1 Fourier, "Theorie Analytique de la Chaleur", Paris, 1822
2 Drude, P., Annalen der Physik, 1, 566 (1900).

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6-1 Molecular Thermal Transfer of Energy Between Two Parallel Isothermal Surfaces

The molecular thermal transfer of energy or molecular conduction of heat between two parallel isothermal surfaces may be described according to the following equation, variously attributed to Fourier and to Newton.

$$\dot{Q} = \frac{dQ}{d\theta} = -KA\frac{dt}{dL} \tag{6-1}$$

This equation may be interpreted as a quantitative statement that the rate of molecular conduction of heat or thermal transfer of energy, $\dot{Q}$, between two parallel elements of surface, each of which is at a constant temperature, and each of which has an area, A, is proportional to the area, A, and to the temperature gradient, $\frac{dt}{dL}$, between the two surfaces. The negative sign signifies that the thermal transfer of energy is in the direction of decreasing temperature.

A sketch of the idea outlined is given in Fig. 6-1. It should be remembered that the thermal conductivity, k, varies with the state of the system.

6-2 Fourier Partial Differential Equation for Molecular Thermal Transfer of Energy or Molecular Conduction of Heat

Equation (6-1) may be written in another form by considering that the net gain of energy of a small element of volume of material raises the temperature of the element. The amount of temperature rise is dependent upon the specific heat and the specific weight of the material.

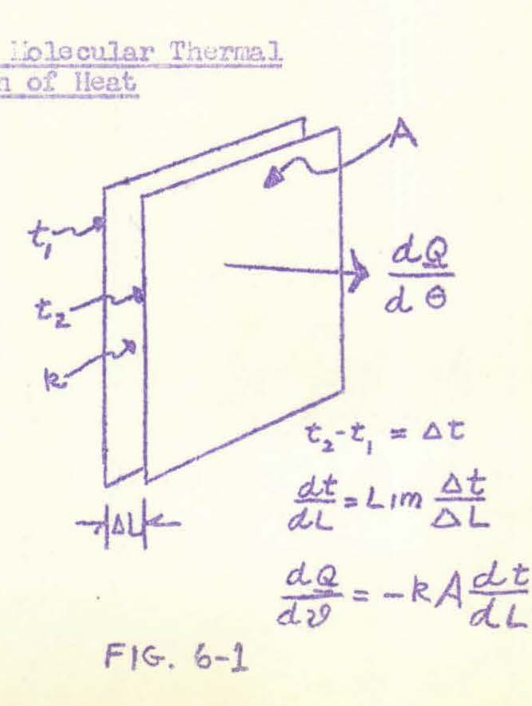


FIG. 6-1

The Fourier partial differential equation for the thermal transfer of energy or molecular conduction of heat which will be derived, may be written as follows:

$$\frac{\partial t}{\partial \theta} = \frac{1}{C_g \sigma}\left\{\frac{\partial}{\partial x}\left(k\frac{\partial t}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial t}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial t}{\partial z}\right)\right\} \qquad (6\text{-}2)$$

It is assumed that the medium is isotropic. This equation does not assume that the thermal conductivity, k, is a constant independent of state and hence of position in the material. If k is not a function of position, equation (6-2) may be written as follows:

$$\frac{\partial t}{\partial \theta} = \frac{k}{C_g \sigma}\left\{\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}+\frac{\partial^2 t}{\partial z^2}\right\} \qquad (6\text{-}3)$$

In equations (6-2) and (6-3) the specific weight of the material is designated by σ and the specific heat, under conditions of restraint, is designated by Cg. It is possible to express Cg in terms of the specific heat at constant pressure, Cp, and the latent heat of pressure change, lp, as follows:

$$C_g = C_p + l_p\left[\frac{dP}{dt}\right]_g \qquad (6\text{-}4)$$

And it is possible, similarly, to express Cg in terms of the specific heat at constant volume, Cv, and the latent heat of volume change, lv, as follows:

$$C_g = C_v + l_v\left[\frac{dV}{dt}\right]_g \qquad (6\text{-}5)$$

A more complete explanation of the terms lp and lv will be found in Lacey and Sage[1] and in Goranson[2].

- -

1 Lacey and Sage, Thermodynamics of One-Component Systems, California Institute of Technology (1941)

2 Goranson, Thermodynamics of Multi-Component Systems, Carnegie Institution of Washington (1930)

Equation (6-2) will be derived by consideration of an element of volume of material shown in Fig. 6-2. (Cartesian coordinates are used although their use is not essential for such relations. The choice of coordinates to solve a given problem would depend upon the type of boundaries encountered in the problem.) The element shown in Fig. 6-2 has edges of respective length dx, dy, and dz, each of which is parallel to a coordinate axis. One corner of the element is located at the point P which has coordinates x, y, and z.

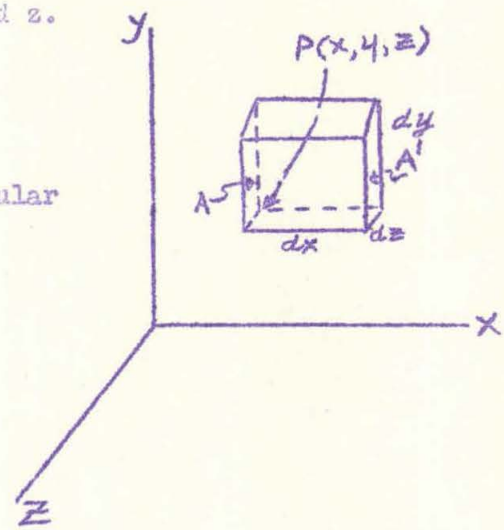


Fig 6-2

The flow of thermally transferred energy into the element, through the face A, which is perpendicular to the X-axis and passes through the point P is, by equation (6-1),

$$-k\frac{dt}{dx}\,dy\,dz \qquad (6\text{-}6)$$

Similarly, the flow of thermally transferred energy into the element through the face A, which is perpendicular to the X-axis but does not pass through P, is

$$\left\{k\frac{dt}{dx}+\frac{d}{dx}\left(k\frac{dt}{dx}\right)dx\right\}dy\,dz \qquad (6\text{-}7)$$

The net addition of energy to the element in unit time through the two faces perpendicular to the X-axis is the sum of the terms given in expressions (6-6) and (6-7):

$$\frac{d}{dx}\left(k\frac{dt}{dx}\,dx\right)dy\,dz \qquad (6\text{-}8)$$

Two other terms may be derived for the net flow of energy into the element through the faces perpendicular, respectively, to the Y-axis and the Z-axis. These terms are,

$$\frac{\partial}{\partial y}\left(k\frac{dt}{dy}\,dy\right)dx\,dz \qquad (6\text{-}9)$$

and, $\frac{\partial}{\partial z}\left(k\frac{\partial t}{\partial z}\,dz\right)dx\,dy$ (6-10)

The net flow of thermally transferred energy into the element is taken as the sum of the flow into the element through all the faces. Adding the terms given in expressions (6-8), (6-9), and (6-10), the following is obtained:

$$\mathring{\underline{q}} = \left[\frac{\partial}{\partial x}\left(k\frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial t}{\partial z}\right)\right]dx\,dy\,dz \qquad (6\text{-}11)$$

The rate of gain of energy of the element may be expressed in terms of the rate of temperature increase, $\frac{\partial t}{\partial \theta}$ the specific heat, Cg, and the specific weight, σ*, as follows:

$$\mathring{\underline{q}} = \frac{\partial t}{\partial \theta}\, C_g \sigma\, dx\,dy\,dz \qquad (6\text{-}12)$$

Equating expressions (6-11) and (6-12), and dividing by dxdydz Cg σ, one obtains:

$$\frac{\partial t}{\partial \theta} = \frac{1}{C_g\sigma}\left\{\frac{\partial}{\partial x}\left(k\frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial t}{\partial z}\right)\right\} \qquad (6\text{-}12.1)$$

This relation reduces to (6-3) when k is not a function of position. The relation (6-2) may be more easily derived by the use of Gauss' or Green's theorem. Such a derivation is given by Carslaw.[1]

---

* Throughout this chapter, the specific weight σ, is employed instead of the density, ρ, because the specific heat is expressed in weight units.

1 Carslaw, H. S., Mathematical Theory of the Conduction of Heat in Solids, Dover Publications, NYC, 1945, p. 9.

A derivation of the analagous case for the electric potential, employing Gauss' theorem, is given by Smythe[2]. The reader is referred to these two sources as giving a fairly representative idea of the solutions which can be obtained to equation (6-2) by analytical means. It is clear from these sources that the class of problems which can be solved analytically, with a reasonable amount of labor, is very restricted.

Several simple cases will be analyzed, however, and analytic solutions presented.

6-3 <u>Conditions Involving Temperature as a Function of One Dimension Only</u>

It is believed desirable to analyze several cases involving the variation of temperature in systems of uniform cross section in which the temperature is a function of only one dimension, since the mathematical solution is rendered less difficult by this stringent restriction.

(i) Thermal Transfer of Energy Between Two Planes, In The Steady State, With Constant Thermal Conductivity.

The steady-state thermal transfer of energy through the material between two parallel planes, each of which is at a constant temperature, may be predicted by a consideration of equation (6-3), if the thermal conductivity, k, is not a function of state. If the X-axis is taken perpendicular to the planes, the temperature is not a function of y or of z. Since steady state was postulated, the temperature is not a function of θ. Hence equation (6-3) reduces to

$$\frac{d^2 t}{d X^2} = 0 \qquad (6\text{-}12.2)$$

---

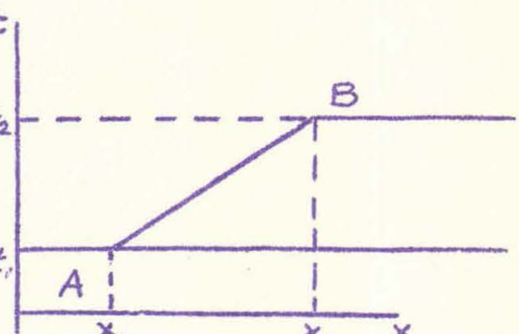
2 Smythe, W. R., Static and Dynamic Electricity, McGraw-Hill Book Company, NYC, 1939, p. 50.

If the temperature at $x = x_1$ is taken as t_1, and that at $x = x_2$ is taken as t_2, then the solution of equation (6-12,2) is

$$t = \frac{x_2 t_1 - x_1 t_2}{x_2 - x_1} + \frac{t_2 - t_1}{x_2 - x_1} x \tag{6-13}$$

as may be verified by substitution into equation (6-12) and the boundary condition equations.

The equation (6-13) is portrayed graphically in Fig. 6-3 as the line AB.



The flow of heat between the planes may be evaluated by the use of equation (6-1) as follows:

$$\dot{Q} = -kA \frac{dt}{dx} \tag{6-14}$$

(6-15)

$$\dot{Q} = \frac{-kA(t_2 - t_1)}{x_2 - x_1}$$

Fig. 6-3

(ii) One-Dimensional Thermal Transfer of Energy Between Two Parallel Planes Maintained at Fixed Temperatures. Initial Temperature Distribution Between The Planes Is Known. The Thermal Conductivity Is Independent of State.

The temperature distribution in a material between two parallel planes in a medium having constant thermal conductivity, each of which is held at a fixed temperature, t_2 and t_1, respectively, can be obtained as a function of position and time from equation (6-3) if the initial temperature is known as a function of position. It will be convenient to designate the quantity $k/c_p\sigma$ as K, the thermometric conductivity. Then the problem may be stated as follows:

$$\frac{\partial t}{\partial \theta} = K \frac{\partial^2 t}{\partial x^2} \qquad \text{when } x_1 < x < x_2 \tag{6-16}$$

$$t = t_1 \quad \text{when} \quad X = X_1 \tag{6-17}$$

$$t = t_2 \quad \text{when} \quad X = X_2 \tag{6-18}$$

$$t = f(X) \quad \text{when} \quad \theta = 0 \tag{6-19}$$

Let $t = t_X + t_{X\theta}$ * (6-20)

Then it can be seen** that if t_X and $t_{X\theta}$ satisfy the following equations, the statement of the problem (6-16) through (6-19) is satisfied.

$$\frac{d^2 t_X}{dX^2} = 0 \quad \text{when} \quad X_1 < X < X_2 \tag{6-21}$$

$$t_X = t_1 \quad \text{when} \quad X = X_1 \tag{6-22}$$

$$t_X = t_2 \quad \text{when} \quad X = X_2 \tag{6-23}$$

$$\frac{\partial t_{X\theta}}{\partial \theta} = K \frac{\partial^2 t_{X\theta}}{\partial X^2} \quad X_1 < X < X_2 \tag{6-24}$$

$$t_{X\theta} = 0 \quad \text{when } X = X_1 \text{ and } X = X_2 \tag{6-25}$$

$$t_{X\theta} = f(X) - t_X \quad \text{when} \quad \theta = 0 \tag{6-26}$$

The solution of equation (6-21) is exactly parallel to that of (6-12).

$$t_X = \frac{X_2 t_1 - X_1 t_2}{X_2 - X_1} + \frac{(t_2 - t_1)X}{X_2 - X_1} \tag{6-13}$$

If the value of the transient contribution to the temperature, $t_{X\theta}$ at $\theta = 0$ were

$$A_n \sin \frac{n\pi}{X_2 - X_1}(X - X_1) \tag{6-27}$$

- -

*t_X is a function of coordinates only, which is the steady state solution. $t_{X\theta}$ is a function of time and of coordinates which approaches zero for long times i.e. represents the transient solution.

**The sum of equations (6-21) and (6-24) is the same condition as (6-16) when (6-20) is considered. The sum of (6-22) and (6-25) is the same condition as (6-17). The sum of (6-23 and (6-25) is the same condition as (6-18). Equation (6-26) is the same condition as (6-19).

Where A_n is a constant and n is a positive integer, then equations (6-24) through (6-2') would be satisfied by

$$A_n \sin\left[\frac{n\pi}{X_2 - X_1}(X - X_1)\right] e^{-\frac{K n^2 \pi^2 \theta}{(X_2 - X_1)^2}} \tag{6-28}$$

It can be shown that $[f(X) - t_X](\text{at } \theta = 0)$ may be expanded in a series of terms having the form of (6-28) if the quantity $[f(X) - t_X]$ satisfies Dirichlet's conditions[1]. It can also be shown that equation (6-26) is satisfied by such a series.

The coefficients of the series, which was first described by Fourier, can be evaluated by a formula which is justified in books on advanced calculus[2,3].

$$A_n = \frac{2}{X_2 - X_1} \int_{X_1}^{X_2} \left[f(X') - t_X(X' - X_1)\right] \sin\frac{n\pi}{X_2 - X_1}(X' - X_1)\, dX' \tag{6-29}$$

The value of $t_X(X' - X_1)$ is obtained from equation (6-13) by substituting $X' - X_1$ for X, to avoid confusion with the limits of integration.

- -

1 A description of Dirichlet's conditions is given by Goursat and Hedrick, A Course in Mathematical Analysis, Ginn and Company, NYC (1904), p. 414. A function which is continuous in the interval and which has a finite number of maxima and minima satisfies the conditions. It is not necessary, however, that the function be continuous; though it must have only a finite number of discontinuities and be bounded.

2 Goursat and Hedrick, p. 413.

3 Woods, Advanced Calculus, Ginn and Company, NYC, (1934) p. 296

The value of the temperature between the planes is then given by, after performing the integrations indicated in (6-29),

$$t = \frac{x_2 t_1 - x_1 t_2}{x_2 - x_1} + \frac{(t_2 - t_1)x}{x_2 - x_1} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{t_2 \cos n\pi - t_1}{n} \cdot \sin\frac{n\pi}{\ell}(x - x_1)\, e^{-k\frac{n^2\pi^2\theta}{(x_2-x_1)^2}} + \frac{2}{x_2 - x_1}\sum_{n=1}^{n=\infty} \sin\frac{n\pi}{x_2 - x_1} \cdot (x - x_1)\, e^{\frac{k n^2\pi^2\theta}{(x_2-x_1)^2}} \int_{x_1}^{x_2} f(x_1')\sin\frac{n\pi}{\ell}(x' - x_1)\, dx_1' \qquad (6\text{-}30)$$

(iii) Thermal Transfer of Energy Between Two Parallel Planes At Variable Temperatures. The Inital Temperature Between the Planes Is Known. The Conductivity Is Constant.

For the case of constant thermal conductivity the temperature distribution in the material between two parallel planes, each of which is subjected to a variable temperature, $\phi_1(\theta)$ and $\phi_2(\theta)$, respectively, can be obtained as a function of position and time from equation (6-3) and Duhamel's Theorem[1]. The problem may be stated as follows:

$$\frac{\partial t}{\partial \theta} = k\frac{\partial^2 t}{\partial x^2} \qquad \text{when} \quad x_1 < x < x_2 \qquad (6\text{-}31)$$

$$t = \phi_1(\tau) \qquad \text{when} \quad x = x_1 \qquad (6\text{-}32)$$

$$t = \phi_2(\tau) \qquad \text{when} \quad x = x_2 \qquad (6\text{-}33)$$

$$t = f(x) \qquad \text{when} \quad \theta = 0 \qquad (6\text{-}34)$$

$$\text{Let } t = t_{x\theta} + t_{x\phi} \qquad (6\text{-}35)$$

---

1 A discussion of Duhamel's Theorem is given by Carslaw, p. 16 et seq.

Then it can be seen* that the conditions (6-31) through (6-34) are satisfied if:

$$\frac{\partial t_{X\theta}}{\partial \theta} = \bar{K}\frac{\partial^2 t_{X\theta}}{\partial X^2} \quad \text{when} \quad X_1 < X < X_2 \qquad (6\text{-}36)$$

$$t_{X\theta} = 0 \quad \text{when } X = X_1, \text{ and when } X = X_2 \qquad (6\text{-}37)$$

$$t_{X\theta} = f(X) \quad \text{when} \quad \theta = 0 \qquad (6\text{-}37a)$$

$$\frac{\partial t_{X\phi}}{\partial t} = \bar{K}\frac{\partial^2 t_{X\phi}}{\partial X^2} \quad \text{when} \quad X_1 < X < X_2 \qquad (6\text{-}38)$$

$$t_{X\phi} = \phi_1(\theta) \quad \text{when} \quad X = X_1 \qquad (6\text{-}39)$$

$$t_{X\phi} = \phi_2(\theta) \quad \text{when} \quad X = X_2 \qquad (6\text{-}40)$$

and $$t_{X\phi} = 0 \quad \text{when} \quad \theta = 0 \qquad (6\text{-}41)$$

It may be seen by consideration** of (6-36) that a solution is

$$t_{X\theta} = \frac{2}{X_2 - X_1}\sum_{n=1}^{n=\infty} e^{\frac{-\bar{K} n^2 \pi^2}{(X_2 - X_1)^2}} \sin\frac{n\pi(X - X_1)}{X_2 - X_1}\int_{X_1}^{X_2} f(X')\sin\frac{n\pi}{X_2 - X_1}(X' - X_1)\,dX' \qquad (6\text{-}42)$$

To obtain $t_{X\phi}$, it is necessary to use Duhamel's Theorem.

- -

* The sum of equations (6-36) and (6-38) is equivalent to equation (6-31). The sum of (6-37) and (6-39) is equivalent to (6-32). The sum of (6-37) and (6-40) is equivalent to (6-33). The sum of (6-37a) and (6-41) is equivalent to (6-35).

** The solution given is arrived at by a separation of variables followed by a choice of constants to fit the prescribed boundary conditions.

One form of the theorem is as follows[1]:

If $t = F(xyz, \theta)$ represents the temperature at $(x\ y, z)$ at the time θ in a solid in which the initial temperature is kept at unity, then the solution of the problem when the surface is kept at temperature $\phi(\theta)$ is given by

$$t = \int_0^t \phi(\theta') \frac{\partial}{\partial \theta} F(x, y, z, \theta - \theta') d\theta' \qquad (6\text{-}43)$$

It can be shown that $F_1(x, X, \theta - \theta')$, which refers to the temperature in the material between planes when the temperature at $x = X_1$ is kept at unity is:

$$F_1(X, \theta - \theta') = 1 - \frac{X - X_1}{X_2 - X_1} - \frac{2}{\pi} \sum_{n=1}^{n=\infty} \frac{1}{n} e^{-\frac{k n^2 \pi^2 (\theta - \theta')}{X_2 - X_1}} \sin \frac{n\pi}{X_2 - X_1}(X - X_1) \qquad (6\text{-}44)$$

Similarly,

$$F_2 = \frac{X - X_1}{X_2 - X_1} + \frac{2}{\pi} \sum_{n=1}^{n=\infty} \frac{1}{n} \cos n\pi\, e^{\frac{-K n^2 \pi^2 (\theta - \theta')}{(X_2 - X_1)^2}} \sin \frac{n\pi(X - X_1)}{X_2 - X_1} \qquad (6\text{-}45)$$

refers to the temperature of the material when the plane $x = x_2$ is kept at unity.

Multiplying (6-44) by $\phi_1(\theta')$, (6-45) by $\phi_2(\theta')$, adding the products, and substituting into (6-43)

$$t_{x\phi} = \int_0^\theta \left[\phi_1(\theta') \frac{\partial}{\partial \theta} F_1(X, \theta - \theta') + \phi_2(\theta') \frac{\partial}{\partial \theta} F_2(X, \theta - \theta')\right] d\theta' \qquad (6\text{-}46)$$

This may be simplified to the following form:

$$t_{x\phi} = \frac{2K\pi}{(X_2 - X_1)^2} \sum_{n=1}^{n=\infty} n e^{\frac{-K n^2 \pi^2}{(X_2 - X_1)^2}} \sin \frac{n\pi(X - X_1)}{X_2 - X_1} \int_0^{\theta'} e^{\frac{K n^2 \pi^2 \theta'}{(X_2 - X_1)^2}} [\phi_1(\theta) - \phi_2(\theta)] d\theta' \qquad (6\text{-}47)$$

Therefore, finally, adding $t_{x\theta}$ and $t_{x\phi}$,

$$t = \frac{2}{X_2 - X_1} \sum_{n=1}^{n=\infty} e^{\frac{-K n^2 \pi^2 \theta}{(X_2 - X_1)^2}} \sin \frac{n\pi(X - X_1)}{X_2 - X_1} \left[\int_{X_1}^{X_2} f(x' - X_1) \sin \frac{n\pi(X' - X_1)}{X_2 - X_1} dX' \right. \qquad (6\text{-}48)$$
$$\left. + \frac{nK\pi}{X_2 - X_1} \int_0^{\theta'} e^{\frac{K n^2 \pi^2 \theta'}{(X_2 - X_1)^2}} [\phi_1(\theta') - (-1)^n \phi_2(\theta')] d\theta' \right]$$

---

1 Carslaw, page 13

(iv) Graphical Solution of Temperature Between Two Parallel Planes Maintained at Fixed Temperatures. The Initial Temperature Distribution is Known.

A graphical method of solution of the problem of determining the temperature between two planes, if the initial temperature distribution is known and if the temperature of each plane is constant has been devised by E. Schmidt[1].

The Fourrier equation for non-steady state flow of heat in 1 dimension may be written as follows:

$$\frac{\partial t}{\partial \theta} = K\frac{\partial^2 t}{\partial x^2} \qquad (6\text{-}16)$$

Let t_{np} refer to a temperature $n\Delta x$ units from the origin after $p\ \Delta\theta$ units of time from the initial state. Then the partial derivative $\frac{\partial t}{\partial \theta}$ evaluated at $n\ \Delta x$ units from the origin and at the time $p\ \Delta\theta$ may be written as the limit, as $\Delta\theta$ approaches zero, of $\frac{t_{n,p+1} - t_{n,p}}{\Delta\theta}$. Similarly, the partial derivative $\frac{\partial t}{\partial x}$ at $n\Delta x$ and $p\Delta\theta$ may be written as the limit, of $\frac{t_{n+1,p} - t_{n,p}}{\Delta x}$, as Δx approaches zero. It follows that the second partial derivative, $\frac{\partial}{\partial x}\left(\frac{\partial t}{\partial x}\right)$ at $n\Delta X$ and $p\ \Delta\theta$ may be written as the limit of $\dfrac{\frac{(t_{n+1,p} - t_{n,p})}{\Delta x} - \frac{(t_{n,p} - t_{n-1,p})}{\Delta x}}{\Delta x}$,

which can also be written as the limit of

$$\frac{t_{n+1,p} + t_{n-1,p} - 2t_{n,p}}{(\Delta x)^2}.$$

The Fourier equation may then be approximated for a volume element having finite dimensions, and for the change of temperature over a finite time interval, as follows:

$$\frac{t_{n,p+1} - t_{n,p}}{\Delta\theta} = \frac{K}{(\Delta x)^2}\left(t_{n+1,p} + t_{n-1,p} - 2t_{n,p}\right) \qquad (6\text{-}49)$$

---

1 Schmidt, E., Festschrift zum Siebzigsten Geburstag August Fopples, Julius Springer, Berlin (1924). See also Euken, A., und Jakob, M., Der Chemie - Ingenieur, Akademische Verlagsgesellschaft, Leipzig (1933).

If the value of $\frac{K\Delta\theta}{(\Delta x)^2}$ is taken as 1/2, then equation (6-49) becomes:

$$t_{n,p+1} = \frac{1}{2}\left(t_{n+1,p} + t_{n-1,p}\right) \qquad (6\text{-}47)$$

This equation states that at the end of the p + 1 th time interval, the temperature at a distance of n ΔX units from the origin is equal to the arithmetic mean of the temperature at a distance of (n-1) ΔX units and of that at a distance of (n+1) ΔX units at the end of the p th time interval. The time interval, $\Delta\theta$, related to the distance interval, ΔX, by the relation given above.

An example of graphical solution of the temperature in a material between two parallel planes is given in Figure 6-4.

The distance between the planes is taken as 4 ΔX. The temperatures of the planes are assumed to be constant, $t_{o,p}$ and $t_{4,p}$, respectively. The limiting value of the temperature (the steady-state solution) will be the straight line drawn between 0, t_p and 4ΔX, $t_{4,p}$. It is trivial, in the one-dimensional case, to observe that the steady-state solution may be approximated by applying the method given to an arbitrary temperature distribution a large number of times.

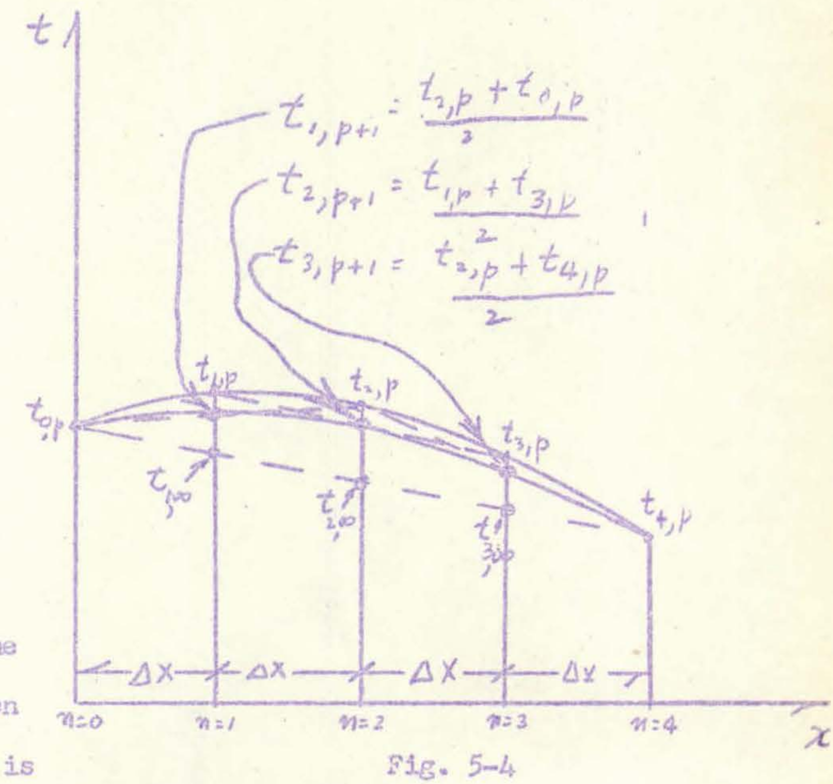
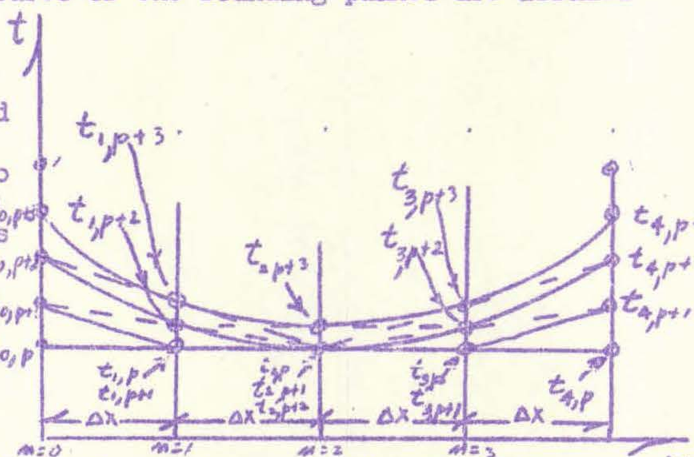


Fig. 5-4

The initial temperature distribution is given at 5 points. The points are designated in Figure 6-4 as $t_{0,p}$, $t_{1,p}$, $t_{2,p}$, $t_{3,p}$, and $t_{4,p}$. The temperature distribution at the time $\Delta\theta$ seconds later may be found at 5 points, $t_{0,p}$, $t_{1,p+1}$, $t_{2,p+1}$, $t_{3,p+1}$, and $t_{4,p+1}$ as indicated in the figure. The quantity $t_{1,p+1}$, for example is found by averaging $t_{0,p}$ and $t_{2,p}$.

(v) Graphical Solution of Temperature Between Two Parallel Planes Maintained at Variable Temperatures. The Initial Temperature Distribution is Known.

An example of graphical solution of the temperatures in a material between two parallel planes is given in Figure 6-5. The temperatures of the bounding planes are assumed to vary with time, in a known manner. The method of solution is similar to that used in the previous section.



d. Condition Involving Temperature as a Function of Two Dimensions.

(i) An Example of Analytical Solution of a Two-Dimensional Thermal Transfer of Energy, Steady State.

Consider the thermal transfer of energy in a semi-infinite slab of material, which extends from $x = 0$ to $x = x_1$, from $y = 0$ to $y = M$ (M is a number which is large compared to x_1) and from $z = -M$ to $z = +M$. A section cut by the plane $z = 0$ is shown in Figure 6-6.

Consider that the sides A and A' (perpendicular to the X-axis as shown in Figure 6-6) are maintained at a temperature $t = 0$*. Consider that the side B is maintained at temperature t_1, and that B' is maintained at temperature $t = 0$. Consider that the sides C and C' (perpendicular to the z-axis) are maintained insulated from the surroundings.

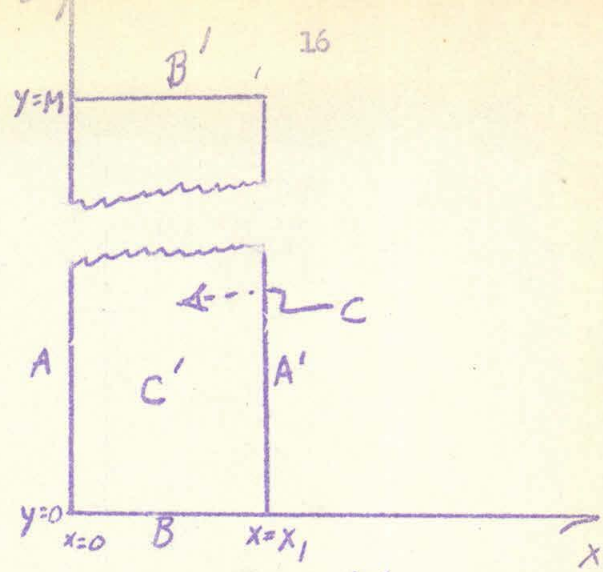


Figure 6-6

Then, since by symmetry the temperature is not a function of z, and since steady state is assumed, equation (6-3) may be written,

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = 0 \qquad (6\text{-}48)$$

It may be seen that a solution of the

$$t = X(x)\,Y(y) \qquad (6\text{-}49)$$

may be employed to replace equation (5-48) by two ordinary differential equations** The quantity $X(x)$ is used to designate a function of x only and the quantity $Y(y)$ is used to designate a function of y only.

The substitution mentioned gives

$$Y\frac{d^2 Y}{dy^2} + X\frac{d^2 X}{dy^2} = 0 \qquad (6\text{-}50)$$

or

$$-\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{Y}\frac{d^2 Y}{dy^2} \qquad (6\text{-}51)$$

---

* No loss of generality occurs by this assumption since absolute temperature does not enter into equation (6-3) if $k =$ constant and $t \neq f(\theta)$.

** The class of partial differential equations which have solutions of this type is very restricted. Linear homogeneous equations with constant coefficients are always solvable by this type of substitution. Linear homogeneous equations with variable coefficients are, sometimes. See Miller, F. H., Partial Differential Equations, Wiley & Sons, NYC, 1941.

Since each side of this equation is a function of a different independent variable and since the equation is identically satisfied for all values of both variables, each side must be equal to a constant, which may be taken as α^2; and

$$\frac{d^2 X}{dx^2} + \alpha^2 X = 0 \qquad (6\text{-}52)$$

and

$$\frac{d^2 Y}{dy^2} - \alpha^2 Y = 0 \qquad (6\text{-}53)$$

The solutions to equations (5-52 and (5-53), respectively are

$$X = \beta_1 e^{i\alpha x} + \beta_1' e^{-i\alpha x} \qquad (6\text{-}54)$$

and

$$Y = \beta_2 e^{\alpha y} + \beta_2' e^{-\alpha y} \qquad (6\text{-}55)$$

Therefore,

$$t = X \cdot Y \qquad (6\text{-}49)$$

or,

$$t = (\beta_1 e^{i\alpha x} + \beta_1' e^{-i\alpha x})(\beta_2 e^{\alpha y} + \beta_2' e^{-\alpha y}) \qquad (6\text{-}56)$$

If the following definitions of $\sin \alpha x$ and $\cos \alpha x$ are employed,

$$\sin \alpha x = (e^{i\alpha x} - e^{-i\alpha x})/2i \qquad (6\text{-}57)$$

$$\cos \alpha x = (e^{i\alpha x} + e^{-i\alpha x})/2 \qquad (6\text{-}58)$$

then equation (5-56) may be written as

$$t = (\beta_1'' \sin \alpha x + \beta_1''' \cos \alpha x)(\beta_2 e^{\alpha y} + \beta_2' e^{-\alpha y}) \qquad (6\text{-}59)$$

In order to h ve $t = 0$ for $x = 0$, for any value of y, $\beta_1''' = 0$. Also to have $t \to 0$ as y becomes large, for any x, $\beta_2 = 0$. Hence, equation (5-59) reduces to

$$t = \beta_1'' \beta_2' e^{-\alpha y} \sin \alpha x \qquad (6\text{-}60)$$

If α is taken as $\frac{n\pi}{x_1}$, where n is an integer, then $t = 0$ for $x = x_1$, for any y. The constant $\beta_1'' \beta_2'$, may be designated by β_5. Therefore,

$$t = \beta_5 e^{-\frac{n\pi}{x_1} y} \sin \frac{n\pi}{x_1} x \qquad (6\text{-}61)$$

The condition that $t = t_1$ at $y = 0$ is not yet fulfilled. It is possible to fulfill this condition by taking a sum of terms of the type given on the right side of equation (6-60). The sum still satisfies the partial differential equation (6-48) since the equation is linear.

It is known* that

$$\frac{\pi}{4} = \sin x' + \frac{1}{3}\sin 3x' + \frac{1}{5}\sin 5x' + \cdots \qquad (6\text{-}62)$$

for $0 < x' < \pi$

Multiplying by $4t_1/\pi$,

$$t_1 = \frac{4t_1}{\pi}\left\{\sin x' + \frac{1}{3}\sin 3x' + \frac{1}{5}\sin 5x' + \cdots\right\} \qquad (6\text{-}63)$$

It may be seen that the boundary condition $t = t_1$ at $y = 0$ for $0 < x < x_1$ and the partial differential equation are satisfied by

$$t = \frac{t_1 4}{\pi}\left\{e^{-\frac{y\pi}{x_1}}\sin\frac{\pi x}{x_1} + e^{-\frac{3y\pi}{x_1}}\sin\frac{3\pi x}{x_1} + e^{-\frac{5y}{x_1}}\sin\frac{5\pi x}{x_1} + \cdots\right\} \qquad (6\text{-}64)$$

A sketch of the lines of constant temperature and of direction of flow of energy is given in Figure 6-7.

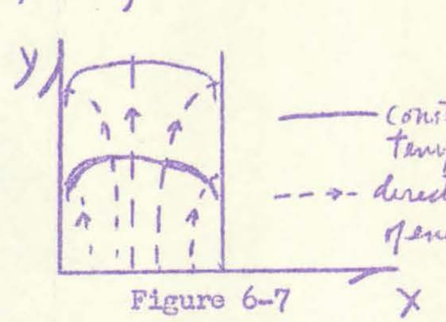


Figure 6-7

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* Peirce, A Short Table of Integrals, Ginn and Co., NYC, Equation 808. (This is derived from the usual Fourier Series.)

(ii) Method of Numerical Solution of Problems in Two-Dimensional Thermal Transfer of Energy. The Initial Temperature Distribution is Known.

The method used in the study of one-dimensional temperature distribution by Schmidt has been applied by him to the study of two-dimensional problems[1].

The Fourier equation for non-steady state flow of heat in two dimensions may be written as follows:

$$\frac{\partial t}{\partial \theta} = k\left(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2}\right) \qquad (6\text{-}65)$$

Let t_{nmk} refer to a temperature $n\Delta x$ units in the x direction and $m\Delta y$ units in the y direction from the origin, after $p\Delta\theta$ units of time from the initial state as before, the partial derivative, $\frac{\partial t}{\partial \theta}$ may be written as the limit of $(t_{n,m,p+1} - t_{n,m,p})/\Delta\theta$ as $\Delta\theta$ approaches zero. Also, as before, the value of $\frac{\partial^2 t}{\partial x^2}$ may be written as the limit of $(t_{n+1,m,p} + t_{n-1,m,p} - 2t_{n,m,p})/(\Delta x)^2$

If the interval distance in the x direction, Δx, is taken equal to the interval distance in the y direction, Δy the value of $\frac{\partial^2 t}{\partial y^2}$ may be written as the limit of

$$(t_{n,m+1,p} + t_{n,m-1,p} - 2t_{n,m,p})/(\Delta x)^2$$

The Fourier equation may then be written, for a volume element having finite dimensions, and for the change of temperature over a finite time interval, as follows:

$$t_{n,m,p+1} - t_{n,m,p} = \frac{k\Delta\theta}{(\Delta x)^2}\left(t_{n+1,m,p} + t_{n-1,m,p} + t_{n,m+1,p} + t_{n,m-1,p} - 4t_{n,m,p}\right) \qquad (6\text{-}66)$$

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1 Schmidt, E., Festschrift zum siebzigsten Gelurstag August Fopples, Julius Springer Berlin (1924).

If the quantity $\frac{K\Delta\vartheta}{(\Delta x)^2}$ is taken as $\frac{1}{4}$ in order that $t_{n,m,p}$ may be eliminated, then equation 6-(66) may be written as follows:

$$(t_{n,m,p+1}) = \frac{1}{4}(t_{n+1,m,p} + t_{n-1,m,p} + t_{n,m+1,p} + t_{n,m-1,p}) \quad (6\text{-}67)$$

It may be seen from equation (6-67) that the temperature at the point $n\Delta x$, $m\Delta y$ at the end of the p+1 th time interval is equal to the arithmetic mean of the temperatures at the points $((n+1)\Delta x, 0)$, $((n-1)\Delta x, 0)$, $(0, (m-1)\Delta y)$, and $(0, (m+1)\Delta y)$ at the end of the pth time interval. It is necessary to know the temperature at all points for one instant and the boundary temperatures for every instant in order to predict the temperature at any later time.

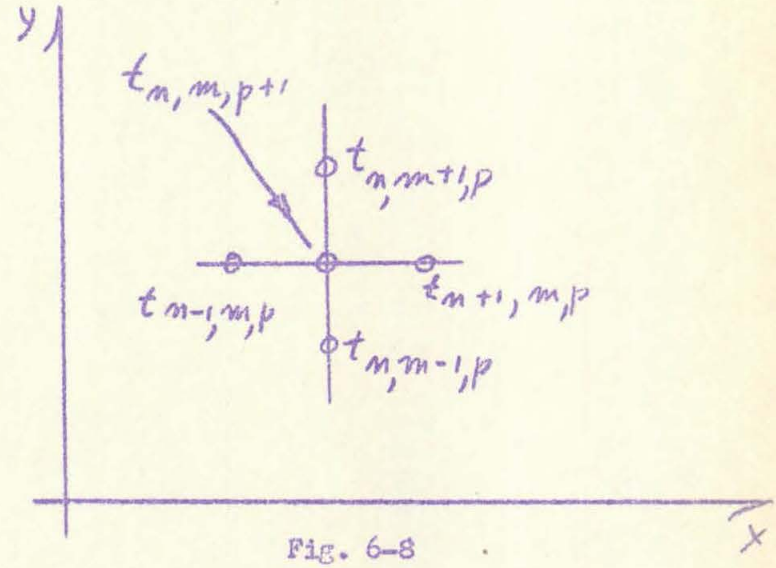

Fig. 6-8